Real Polynomials - Practice Test 6

10 questions | 58 marks | 75 minutes

Combine coefficient constraints, repeated-root arguments and parameter-dependent real roots.

Equivalent correct methods earn the corresponding marks.

1. $P_a(x) = (x+a)^3 - x^3$. [3 marks]

(a) Expand P and identify its degree when $a \neq 0$. [2 marks]

The cubic terms cancel, while $3a \neq 0$ is the quadratic coefficient.

Answer: $P = 3ax^2 + 3a^2x + a^3$; degree 2.

(b) What happens when $a = 0$? [1 mark]

All three remaining coefficients are zero.

Answer: P is the zero polynomial, with no ordinary degree.

2. Consider the monic cubic with factors $(x-a)(x-2a)(x+3a)$. [6 marks]

(a) Expand it in terms of a. [3 marks]

First multiply $(x-a)(x-2a) = x^2 - 3ax + 2a^2$, then multiply by $x+3a$.

Answer: $x^3 - 7a^2x + 6a^3$.

(b) Hence find a and all roots if the polynomial equals $x^3 - 28x + 48$. [3 marks]

$a^2 = 4$ and $a^3 = 8$; the second condition fixes the sign.

Answer: $a = 2$; roots $2, 4, -6$.

3. $P(x) = x^3 + ax^2 + bx + 5$ has remainder 3 on division by $x - 1$ and remainder 9 on division by $x + 1$. [6 marks]

(a) Find a and b. [3 marks]

$a + b = -3$ and $a - b = 5$, giving $a = 1, b = -4$.

Answer: $a = 1, b = -4$.

(b) Find the remainder on division by $x^2 - 1$. [3 marks]

For $R = mx + c$, $m + c = 3$, $-m + c = 9$, so $m = -3, c = 6$.

Answer: $-3x + 6$.

4. P is divided by $(x - 1)^2$, leaving remainder $mx + c$. It is known that $P(1) = 3$ and $P'(1) = 4$. [6 marks]

(a) Use the division identity and its derivative to find m and c. [4 marks]

$P = (x - 1)^2 Q + mx + c$. At 1, $m + c = 3$. Differentiating and setting $x = 1$ gives $m = 4$.

Answer: $m = 4, c = -1$; remainder $4x - 1$.

(b) Does this determine the entire polynomial P? [2 marks]

Any multiple of $(x - 1)^2$ may be added without changing either supplied condition.

Answer: No.

5. $P(x) = x^4 + ax^3 + bx^2 + ax + 1$ has zeros 2 and -1. [7 marks]

(a) Find a and b. [4 marks]

$P(2) = 17 + 10a + 4b = 0$ and $P(-1) = 2 - 2a + b = 0$. Substitute $b = 2a - 2$ to get $a = -\frac{1}{2}$.

Answer: $a = -\frac{1}{2}, b = -3$.

(b) Show that all real roots are $2, \frac{1}{2}, -1$, with -1 repeated. [3 marks]

Expand the stated product or divide by the known factors to obtain the remaining quadratic.

Answer: $2P = (x + 1)^2(2x - 1)(x - 2)$.

6. $P_k(x) = x^4 - 2kx^2 + 1$, where k is real. [7 marks]

(a) Solve for x^2 in terms of k. [3 marks]

Use the quadratic formula after setting $u = x^2$.

Answer: $x^2 = k \pm \sqrt{k^2 - 1}$ when $|k| \geq 1$.

(b) State the number of distinct real roots for $k < 1$, $k = 1$ and $k > 1$. [4 marks]

For $-1 < k < 1$ the u-roots are non-real; for $k \leq -1$ they are negative. At $k = 1$, $u = 1$; for $k > 1$, both u-values are positive and distinct.

Answer: Zero, two, four respectively.

7. Consider the equations $x^3 - 3x = c$ for varying real c. [6 marks]

(a) Find the local maximum and minimum of $x^3 - 3x$. [3 marks]

The derivative is $3x^2 - 3$; classify using the derivative sign or second derivative.

Answer: Maximum $(-1, 2)$ and minimum $(1, -2)$.

(b) How many distinct real solutions are there when $|c| < 2$, $|c| = 2$, and $|c| > 2$? [3 marks]

Compare the horizontal line $y = c$ with the three monotonic parts separated by the turning points; tangency at an extreme creates a repeated root.

Answer: Three, two, one respectively.

8. Let $P(x) = (x + 2)(x - 1)^2$ and $Q(x) = 2P(3 - x)$. [5 marks]

(a) Find the factorised form and roots of Q. [3 marks]

Replace every occurrence of x in P by $3 - x$.

Answer: $Q(x) = 2(5 - x)(2 - x)^2$; roots 5 simple and 2 double.

(b) Find its leading coefficient and y-intercept. [2 marks]

The leading factors give $2(-x)(x^2) = -2x^3$; $Q(0) = 2(5)(4) = 40$.

Answer: Leading coefficient -2; y-intercept 40.

9. A sensor calibration is modelled by a polynomial P of degree at most 3. It reports $P(0)=1$, $P(1)=2$, $P(2)=9$, $P(3)=28$. [6 marks]

(a) Find a polynomial fitting all four values. [4 marks]

Writing ax^3+bx^2+cx+d gives $d=1$, then $a+b+c=1$, $4a+2b+c=4$, $9a+3b+c=9$; solve to get $a=1, b=c=0$.

Answer: $P(x)=x^3+1$.

(b) Explain why no other polynomial of degree at most 3 can fit the same data. [2 marks]

The degree bound forces that difference to be the zero polynomial.

Answer: The difference would have four distinct zeros but degree at most 3.

10. A polynomial P has the form $P(x)=(x-2)Q(x)+7$. [6 marks]

(a) Find the remainder of $P(x)^2-14P(x)$ on division by $x-2$. [3 marks]

At $x=2$, $P=7$; hence $7^2-14(7)=-49$.

Answer: -49.

(b) Prove that $P(x)^2-14P(x)+49$ is divisible by $(x-2)^2$. [3 marks]

The given division identity makes $P-7=(x-2)Q$; square both sides. This proves divisibility by the squared factor, not merely by one copy.

Answer: It equals $(P(x)-7)^2=(x-2)^2Q(x)^2$.