Real Polynomials - Practice Test 3

10 questions | 51 marks | 65 minutes

Apply polynomial models to geometry, intersections and reconstruction problems.

Equivalent correct methods earn the corresponding marks.

1. Let $P(x) = x^3 + 2x^2 - x - 2$. [4 marks]

(a) Find $P(0)$ and $P(1)$. [1 mark]

Substitute into the original rule.

Answer: -2 and 0.

(b) Factor P completely and list its real roots. [3 marks]

Group as $x^2(x+2) - (x+2)$.

Answer: $(x+2)(x-1)(x+1)$; roots $-2, -1, 1$.

2. A rectangle has side lengths $x+2$ and $x+5$ metres, where $x > 0$. [5 marks]

(a) Expand its area rule. [2 marks]

Multiply the side expressions.

Answer: $A = x^2 + 7x + 10$.

(b) Find x when the area is 28 m^2. [3 marks]

$x^2 + 7x - 18 = (x+9)(x-2) = 0$; reject -9 because $x > 0$.

Answer: $x = 2$ m.

3. A monic cubic $P(x) = x^3 + ax^2 + bx + 2$ has zeros 1 and 2. [5 marks]

(a) Find a and b. [3 marks]

$a + b = -3$ from $P(1) = 0$; $2a + b = -5$ from $P(2) = 0$.

Answer: $a = -2, b = -1$.

(b) Find the remaining root. [2 marks]

$P = (x - 1)(x - 2)(x + 1)$.

Answer: -1.

4. For $P(x) = 2x^3 + 3x^2 - 8x + 3$, consider the proposed factor $2x - 1$. [5 marks]

(a) Prove it is a factor. [2 marks]

$\frac{1}{4} + \frac{3}{4} - 4 + 3 = 0$.

Answer: $P(\frac{1}{2}) = 0$.

(b) Find the full real factorisation. [3 marks]

Division gives $x^2 + 2x - 3 = (x + 3)(x - 1)$.

Answer: $(2x - 1)(x + 3)(x - 1)$.

5. The curve $y = x^3 - 4x$ meets the line $y = 2x$. [5 marks]

(a) Form and solve the equation for their x-coordinates. [3 marks]

Equate the y-values, then move $2x$ to the left.

Answer: $x(x^2 - 6) = 0$; $x = 0, \pm\sqrt{6}$.

(b) State all intersection points. [2 marks]

Substitute the three x-values into $y = 2x$.

Answer: $(0, 0), (\sqrt{6}, 2\sqrt{6}), (-\sqrt{6}, -2\sqrt{6})$.

6. A student solves $(x-1)(x^2-4)=0$ by cancelling $x-1$. [4 marks]

(a) Explain the error. [2 marks]

A factor can be zero; it cannot be divided out without checking that case.

Answer: It assumes $x \neq 1$, so can remove a valid solution.

(b) Give the complete solution set. [2 marks]

Set each linear factor in $(x-1)(x-2)(x+2)$ equal to zero.

Answer: $\{-2, 1, 2\}$.

7. $P(x)=(x+1)(x-2)$ and $Q(x)=-3P(x-1)$. [4 marks]

(a) Write Q in factorised form and find its zeros. [3 marks]

$P(x-1)=((x-1)+1)((x-1)-2)=x(x-3)$.

Answer: $Q=-3x(x-3)$; zeros $0, 3$.

(b) Describe the end behaviour of Q. [1 mark]

The leading term is $-3x^2$.

Answer: Both ends go downwards.

8. Consider $f(x)=x^3-3x^2+2$. [7 marks]

(a) Find and classify the stationary points. [4 marks]

$f'=3x(x-2)$ and $f''=6x-6$.

Answer: $(0, 2)$ maximum and $(2, -2)$ minimum.

(b) Show that its three roots are $1, 1-\sqrt{3}, 1+\sqrt{3}$. [3 marks]

Check $f(1)=0$, divide, then solve the remaining quadratic.

Answer: $f=(x-1)(x^2-2x-2)$.

9. An open box is formed by cutting squares of side x cm from a 12 cm by 6 cm sheet. [6 marks]

(a) Form the volume model and physical domain. [2 marks]

All three resulting dimensions must be positive.

Answer: $V = x(12 - 2x)(6 - 2x);\ 0 < x < 3.$

(b) For volume 40 cm^3, use $x^3 - 9x^2 + 18x - 10 = (x - 1)(x^2 - 8x + 10)$ to find all possible cuts. [4 marks]

The algebraic roots are $1, 4 \pm \sqrt{6}$. Only 1 and $4 - \sqrt{6}$ lie between 0 and 3.

Answer: $x = 1$ cm and $x = 4 - \sqrt{6}$ cm.

10. Design a real polynomial graph of the least possible degree that crosses at -1 and 1, touches at 2, and passes through $(0, -8)$. [6 marks]

(a) Find a factorised rule. [4 marks]

Two simple roots and one double root require degree at least 4. Write the product times a and use $-4a = -8$.

Answer: $P(x) = 2(x + 1)(x - 1)(x - 2)^2.$

(b) State its degree, leading coefficient and end behaviour. [2 marks]

The factor degrees add to 4 and the coefficient is positive.

Answer: Degree 4; leading coefficient 2; both ends up.