Real Polynomials - Practice Test 2

10 questions | 45 marks | 65 minutes

Use identities, remainders and repeated roots to reason about polynomial structure.

Equivalent correct methods earn the corresponding marks.

1. Classify and simplify carefully. [4 marks]

(a) Which are polynomials in x: $x^3+\sqrt{2}$, $x^{-1}+4$, $\frac{x^2-1}{x-1}$ as originally written? [3 marks]

The third agrees with $x+1$ only where $x\neq 1$, so the original function is not the polynomial function on all reals.

Answer: The first is a polynomial; the second is not; the third is a rational expression with a hole at 1.

(b) Simplify $(2x^2+3x-1)+(x^3-5x+4)$. [1 mark]

Collect matching powers.

Answer: x^3+2x^2-2x+3

2. A cubic has zeros $-1, 2, 3$ and $P(0)=12$. [4 marks]

(a) Find its factorised form. [3 marks]

Write $a(x+1)(x-2)(x-3)$; at zero, $6a=12$.

Answer: $P(x)=2(x+1)(x-2)(x-3)$.

(b) Find its leading coefficient and y-intercept. [1 mark]

The factorised form is degree three with leading coefficient 2.

Answer: 2; $(0, 12)$.

3. $P(x) = ax^2 + bx + c$ satisfies $P(0) = 2$, $P(1) = 6$, $P(-1) = 4$. [5 marks]

(a) Find a, b, c. [3 marks]

$c = 2$; $a + b = 4$ and $a - b = 2$, so $a = 3, b = 1$.

Answer: $a = 3, b = 1, c = 2$.

(b) Explain why these conditions determine a unique polynomial of degree at most 2. [2 marks]

Only the zero polynomial can do that.

Answer: The difference of two candidates would have three distinct zeros but degree at most 2.

4. A division gives $D(x) = x^2 + 1$, quotient $Q(x) = 2x - 3$ and remainder $R(x) = x + 4$. [3 marks]

(a) Reconstruct the dividend. [2 marks]

Expand $(x^2 + 1)(2x - 3) + x + 4$.

Answer: $P(x) = 2x^3 - 3x^2 + 3x + 1$.

(b) Explain why the stated remainder has an allowable degree. [1 mark]

A division remainder must have strictly lower degree.

Answer: It is degree 1, less than divisor degree 2.

5. $P(2)=7$ and $P(-1)=1$. [5 marks]

(a) Find the remainder when P is divided by $(x-2)(x+1)$. [3 marks]

Let $R=mx+c$. Then $2m+c=7$ and $-m+c=1$, giving $m=2, c=3$.

Answer: $R(x)=2x+3$.

(b) Can these data determine the whole polynomial? Explain. [2 marks]

Any multiple of $(x-2)(x+1)$ can be added without changing the two values.

Answer: No.

6. Let $P(x)=x^3-3x^2+4$. [5 marks]

(a) Show that 2 is a repeated root. [3 marks]

Check $P(2)=0$, divide by $x-2$, and factor the quotient x^2-x-2.

Answer: $P=(x-2)^2(x+1)$.

(b) State the sign of P immediately on either side of 2. [2 marks]

The square is positive and $x+1>0$ near 2.

Answer: Positive on both sides sufficiently near 2.

7. Factor x^3+x^2+x+1. [4 marks]

(a) Give a complete real factorisation. [2 marks]

Group $x^2(x+1)+(x+1)$.

Answer: $(x+1)(x^2+1)$.

(b) State all real roots and explain why the quadratic gives none. [2 marks]

$x^2+1>0$ for all real x.

Answer: $x=-1$ only.

8. For $f(x) = x^3 - 3x$, use calculus to refine the sketch. [6 marks]

(a) Find the stationary points. [3 marks]

$f' = 3x^2 - 3 = 0$ gives $x = \pm 1$; substitute into f.

Answer: $(-1, 2)$ and $(1, -2)$.

(b) Classify them and state the real zeros. [3 marks]

$f'' = 6x$ gives the types; factor $x(x^2 - 3)$.

Answer: Local maximum at $(-1, 2)$; minimum at $(1, -2)$; zeros $-\sqrt{3}, 0, \sqrt{3}$.

9. Solve $x^4 + x^2 - 6 = 0$ over the reals. [4 marks]

(a) Use a substitution to factor. [2 marks]

Let $u = x^2$ and factor $u^2 + u - 6$.

Answer: $(x^2 + 3)(x^2 - 2) = 0$.

(b) Explain why there are only two real solutions. [2 marks]

A real square is non-negative.

Answer: $x = \pm\sqrt{2}$; $x^2 = -3$ has no real solution.

10. Consider $P_c(x) = (x - 1)^2 + c$. [5 marks]

(a) Find the real roots in terms of c when they exist. [3 marks]

$(x - 1)^2 = -c$ requires $-c \geq 0$.

Answer: $x = 1 \pm \sqrt{-c}$ for $c \leq 0$.

(b) State the number of distinct real roots for $c < 0$, $c = 0$ and $c > 0$. [2 marks]

The two signs give distinct roots only when $\sqrt{-c} > 0$.

Answer: Two, one, zero respectively.