Vector calculus - Test 6

Name: _____ Date: _____

10 questions | 67 marks | 90 minutes

Synthesis, proofs and unfamiliar but hand-solvable applications. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

A scientific calculator may be used.

1. A student replaces the journey $\mathbf{r}(t) = (\cos t, \sin t)$, $0 \le t \le 4\pi$, by the statement "the object moves on $x^2 + y^2 = 1$". [5 marks]

(a) State three pieces of journey information that the Cartesian equation does not specify. [3 marks]

(b) Find the distance travelled and displacement in the original journey. [2 marks]

2. A constant-velocity particle has position $\mathbf{r}(t) = (a + bt, c + dt)$ metres. Measurements give $\mathbf{r}(1) = (3, 2)$ and $\mathbf{r}(3) = (7, -4)$. Time is in seconds. [6 marks]

(a) Determine a, b, c and d. [4 marks]

(b) Find when and where the particle crosses the x-axis. [2 marks]

3. A particle has position $\mathbf{r}(t) = (t^2 - 2t, t^3 - 3t)$ for $0 \leq t \leq 2$. [8 marks]

(a) Find when it is at rest, its position then, and its acceleration. [4 marks]

(b) By writing $t = 1 + h$, find the limiting secant direction at this stationary point and hence its tangent line. [4 marks]

4. A symmetric cubic Bézier arch has controls $P_0 = (0, 0)$, $P_1 = (0, h)$, $P_2 = (3, h)$, $P_3 = (3, 0)$, where $h > 0$. Use

$$\mathbf{B}(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u) P_2 + u^3 P_3, \qquad 0 \le u \le 1.$$

[7 marks]

(a) Show that the greatest height is $\frac{3h}{4}$, and determine h for a required height of $\frac{3}{2}$. [4 marks]

(b) For this value of h, find the two endpoint derivatives and explain the endpoint tangent directions. [3 marks]

5. A particle has acceleration $\mathbf{a}(t) = (0, -2, 4t)$ m/s^2, with $\mathbf{v}(0) = (2, 3, 0)$ m/s and $\mathbf{r}(0) = (1, 0, -1)$ m. [7 marks]

(a) Find the velocity and position functions. [4 marks]

(b) Find its position and speed at $t = 1$. [3 marks]

6. A twice-differentiable motion satisfies $\mathbf{r}(t) \cdot \mathbf{r}(t) = R^2$ throughout an interval, where $R > 0$ is constant. Write $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{r}''$. [7 marks]

(a) Prove that $\mathbf{r} \cdot \mathbf{v} = 0$ and $\mathbf{r} \cdot \mathbf{a} = -|\mathbf{v}|^2$. [4 marks]

(b) A student concludes that all such motions must have constant speed. Give a counterexample and justify why it meets the given condition. [3 marks]

7. A spatial curve is $\mathbf{r}(t) = (e^t, e^{-t}, t)$. [6 marks]

(a) Find its tangent line at $t = 0$, using a new parameter λ. [3 marks]

(b) Find the point on the tangent with $z = 2$. Explain why this point is not the point on the curve with $z = 2$. [3 marks]

8. A drone follows $\mathbf{r}_D(t) = (t, 2t, t^2)$ metres. An escort starts at $(4, 0, 4)$ m at $t = 0$ and must use a constant velocity to meet the drone at $t = 2$ seconds. [7 marks]

(a) Determine the required escort velocity and its position function for $0 \leq t \leq 2$. [4 marks]

(b) Does this meeting also give equal velocities? Calculate both velocities at the meeting. [3 marks]

9. A planar curve is

$\mathbf{r}(t) = (t, \ln(\sec t)), \qquad 0 \leq t \leq \frac{\pi}{3}.$

You may use

$\frac{d}{dt}\ln(\sec t) = \tan t$

and

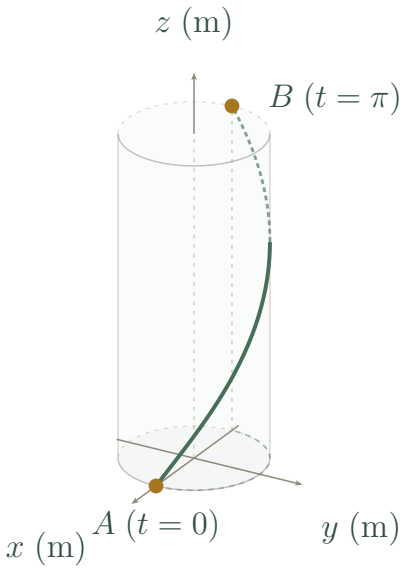
$\int \sec t \, dt = \ln|\sec t + \tan t| + C.$

[6 marks]

(a) Show that the arc-length integrand is $\sec t$ on this interval. [3 marks]

(b) Find the exact curve length. [3 marks]

10. A cable follows $\mathbf{r}(t) = (2\cos t, 2\sin t, 3t)$ metres, $0 \le t \le \pi$. Its endpoints are A at $t = 0$ and B at $t = \pi$. [8 marks]



(a) Find the cable length and the straight-line distance between A and B. [5 marks]

(b) Prove algebraically that the cable length exceeds the endpoint separation. State why comparing squares is valid. [3 marks]