Vector calculus - Test 6

10 questions | 67 marks | 90 minutes

Synthesis, proofs and unfamiliar but hand-solvable applications. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

Equivalent correct methods earn the corresponding marks.

1. A student replaces the journey $\mathbf{r}(t) = (\cos t, \sin t)$, $0 \le t \le 4\pi$, by the statement "the object moves on $x^2 + y^2 = 1$". [5 marks]

(a) State three pieces of journey information that the Cartesian equation does not specify. [3 marks]

The original start is $(1, 0)$; the Cartesian equation does not select a start. [1 mark]

It does not specify anticlockwise rather than clockwise motion. [1 mark]

It does not record the two full turns for $0 \le t \le 4\pi$. [1 mark]

Answer: It omits the starting point, the anticlockwise direction and the fact that the circle is traced twice.

(b) Find the distance travelled and displacement in the original journey. [2 marks]

$|\mathbf{r}'| = 1$, so distance is $\int_0^{4\pi} 1 \, dt = 4\pi$. [1 mark]

The final point equals the initial point, so displacement is $\mathbf{0}$. [1 mark]

Answer: Distance 4π units; displacement $\mathbf{0}$.

2. A constant-velocity particle has position $\mathbf{r}(t) = (a + bt, c + dt)$ metres. Measurements give $\mathbf{r}(1) = (3, 2)$ and $\mathbf{r}(3) = (7, -4)$. Time is in seconds. [6 marks]

(a) Determine a, b, c and d. [4 marks]

$a + b = 3$ and $a + 3b = 7$ give $b = 2$, $a = 1$. [2 marks]

$c + d = 2$ and $c + 3d = -4$ give $d = -3$, $c = 5$. [2 marks]

Answer: $a = 1$, $b = 2$, $c = 5$, $d = -3$.

(b) Find when and where the particle crosses the x-axis. [2 marks]

$y = 5 - 3t = 0$ gives $t = \frac{5}{3}$. [1 mark]

$x = 1 + 2(\frac{5}{3}) = \frac{13}{3}$, so the point is $(\frac{13}{3}, 0)$. [1 mark]

Answer: $t = \frac{5}{3}$ s; point $(\frac{13}{3}, 0)$ m.

3. A particle has position $\mathbf{r}(t) = (t^2 - 2t, t^3 - 3t)$ for $0 \le t \le 2$. [8 marks]

(a) Find when it is at rest, its position then, and its acceleration. [4 marks]

$\mathbf{v} = (2t - 2, 3t^2 - 3)$, so both components vanish at $t = 1$ only in the interval. [2 marks]

$\mathbf{r}(1) = (-1, -2)$. [1 mark]

$\mathbf{a} = (2, 6t)$, so $\mathbf{a}(1) = (2, 6)$. [1 mark]

Answer: At $t = 1$: position $(-1, -2)$ and acceleration $(2, 6)$.

(b) By writing $t = 1 + h$, find the limiting secant direction at this stationary point and hence its tangent line. [4 marks]

$\mathbf{r}(1 + h) - \mathbf{r}(1) = (h^2, 3h^2 + h^3)$. [1 mark]

For $h \neq 0$, this is $h^2(1, 3 + h)$, so the secant direction tends to $(1, 3)$. [2 marks]

Through $(-1, -2)$, the tangent is $y + 2 = 3(x + 1)$. [1 mark]

Answer: Direction $(1, 3)$; tangent $y + 2 = 3(x + 1)$.

4. A symmetric cubic Bézier arch has controls $P_0 = (0, 0)$, $P_1 = (0, h)$, $P_2 = (3, h)$, $P_3 = (3, 0)$, where $h > 0$. Use

$\mathbf{B}(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u)P_2 + u^3 P_3, \qquad 0 \le u \le 1.$

[7 marks]

(a) Show that the greatest height is $\frac{3h}{4}$, and determine h for a required height of $\frac{3}{2}$. [4 marks]

$y = 3hu(1-u)^2 + 3hu^2(1-u) = 3hu(1-u)$. [1 mark]

This downward-opening quadratic has maximum at $u = \frac{1}{2}$, with height $\frac{3h}{4}$. [2 marks]

$\frac{3h}{4} = \frac{3}{2}$ gives $h = 2$. [1 mark]

Answer: Greatest height $\frac{3h}{4}$ at $u = \frac{1}{2}$; $h = 2$.

(b) For this value of h, find the two endpoint derivatives and explain the endpoint tangent directions. [3 marks]

$\mathbf{B}'(0) = 3(P_1 - P_0) = (0, 6)$. [1 mark]

$\mathbf{B}'(1) = 3(P_3 - P_2) = (0, -6)$. [1 mark]

Both are nonzero vertical vectors; the start direction is up and the final direction is down. [1 mark]

Answer: $\mathbf{B}'(0) = (0, 6)$ and $\mathbf{B}'(1) = (0, -6)$; both tangents are vertical, with opposite forward directions.

5. A particle has acceleration $\mathbf{a}(t) = (0, -2, 4t)$ m/s^2, with $\mathbf{v}(0) = (2, 3, 0)$ m/s and $\mathbf{r}(0) = (1, 0, -1)$ m. [7 marks]

(a) Find the velocity and position functions. [4 marks]

Integration and the velocity initial condition give $\mathbf{v} = (2, 3 - 2t, 2t^2)$. [2 marks]

Integration and the position initial condition give $\mathbf{r} = (1 + 2t, 3t - t^2, \frac{2}{3}t^3 - 1)$. [2 marks]

Answer: $\mathbf{v} = (2, 3 - 2t, 2t^2)$; $\mathbf{r} = (1 + 2t, 3t - t^2, \frac{2}{3}t^3 - 1)$.

(b) Find its position and speed at $t = 1$. [3 marks]

$\mathbf{r}(1) = (3, 2, \frac{-1}{3})$. [1 mark]

$\mathbf{v}(1) = (2, 1, 2)$. [1 mark]

The speed is $\sqrt{4 + 1 + 4} = 3$ m/s. [1 mark]

Answer: Position $(3, 2, \frac{-1}{3})$ m; speed 3 m/s.

6. A twice-differentiable motion satisfies $\mathbf{r}(t) \cdot \mathbf{r}(t) = R^2$ throughout an interval, where $R > 0$ is constant. Write $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{r}''$. [7 marks]

(a) Prove that $\mathbf{r} \cdot \mathbf{v} = 0$ and $\mathbf{r} \cdot \mathbf{a} = -|\mathbf{v}|^2$. [4 marks]

Differentiate $\mathbf{r} \cdot \mathbf{r} = R^2$ to obtain $2\mathbf{r} \cdot \mathbf{v} = 0$. [2 marks]

Differentiate $\mathbf{r} \cdot \mathbf{v} = 0$: $\mathbf{v} \cdot \mathbf{v} + \mathbf{r} \cdot \mathbf{a} = 0$. [1 mark]

Thus $\mathbf{r} \cdot \mathbf{a} = -|\mathbf{v}|^2$. [1 mark]

Answer: $\mathbf{r} \cdot \mathbf{v} = 0$; $\mathbf{r} \cdot \mathbf{a} = -|\mathbf{v}|^2$.

(b) A student concludes that all such motions must have constant speed. Give a counterexample and justify why it meets the given condition. [3 marks]

Choose $\mathbf{r}(t) = (\cos t^2, \sin t^2)$ on $1 \le t \le 2$. [1 mark]

Its squared magnitude is 1 at every time, so it meets the condition with $R = 1$. [1 mark]

Its speed is $2t$, which is not constant. [1 mark]

Answer: $\mathbf{r}(t) = (\cos t^2, \sin t^2)$ for $1 \le t \le 2$ has radius 1 but speed $2t$.

7. A spatial curve is $\mathbf{r}(t) = (e^t, e^{-t}, t)$. [6 marks]

(a) Find its tangent line at $t = 0$, using a new parameter λ. [3 marks]

$\mathbf{r}(0) = (1, 1, 0)$. [1 mark]

$\mathbf{r}'(t) = (e^t, -e^{-t}, 1)$, giving $(1, -1, 1)$ at zero. [1 mark]

The tangent is $\mathbf{x} = (1, 1, 0) + \lambda(1, -1, 1)$. [1 mark]

Answer: $\mathbf{x} = (1, 1, 0) + \lambda(1, -1, 1)$, $\lambda \in \mathbb{R}$.

(b) Find the point on the tangent with $z = 2$. Explain why this point is not the point on the curve with $z = 2$. [3 marks]

On the tangent, $\lambda = 2$ gives $(3, -1, 2)$. [1 mark]

On the curve, $z = t = 2$, giving $(e^2, e^{-2}, 2)$. [1 mark]

The coordinates differ; a tangent gives local direction rather than the full curved path. [1 mark]

Answer: Tangent point $(3, -1, 2)$; curve point $(e^2, e^{-2}, 2)$ is different.

8. A drone follows $\mathbf{r}_D(t) = (t, 2t, t^2)$ metres. An escort starts at $(4, 0, 4)$ m at $t = 0$ and must use a constant velocity to meet the drone at $t = 2$ seconds. [7 marks]

(a) Determine the required escort velocity and its position function for $0 \le t \le 2$. [4 marks]

The meeting point is $\mathbf{r}_D(2) = (2, 4, 4)$. [1 mark]

The escort displacement must be $(-2, 4, 0)$ in 2 s. [1 mark]

Its constant velocity is $(-1, 2, 0)$ m/s. [1 mark]

Hence $\mathbf{r}_E(t) = (4 - t, 2t, 4)$ for $0 \le t \le 2$. [1 mark]

Answer: $\mathbf{v}_E = (-1, 2, 0)$ m/s; $\mathbf{r}_E(t) = (4 - t, 2t, 4)$ m.

(b) Does this meeting also give equal velocities? Calculate both velocities at the meeting. [3 marks]

$\mathbf{v}_D = (1, 2, 2t)$, so $\mathbf{v}_D(2) = (1, 2, 4)$. [1 mark]

The escort still has velocity $(-1, 2, 0)$. [1 mark]

These are unequal; equal positions at one time do not imply equal velocities. [1 mark]

Answer: No. Drone velocity $(1, 2, 4)$ m/s; escort velocity $(-1, 2, 0)$ m/s.

9. A planar curve is

$\mathbf{r}(t) = (t, \ln(\sec t)), \qquad 0 \le t \le \frac{\pi}{3}.$

You may use

$\frac{d}{dt}\ln(\sec t) = \tan t$

and

$\int \sec t \, dt = \ln|\sec t + \tan t| + C.$

[6 marks]

(a) Show that the arc-length integrand is $\sec t$ on this interval. [3 marks]

$\mathbf{r}' = (1, \tan t)$, so its magnitude is $\sqrt{1 + \tan^2 t}$. [1 mark]

The identity $1 + \tan^2 t = \sec^2 t$ gives $|\sec t|$. [1 mark]

Since $\cos t > 0$ on $[0, \frac{\pi}{3}]$, this equals $\sec t$. [1 mark]

Answer: $|\mathbf{r}'(t)| = \sec t$.

(b) Find the exact curve length. [3 marks]

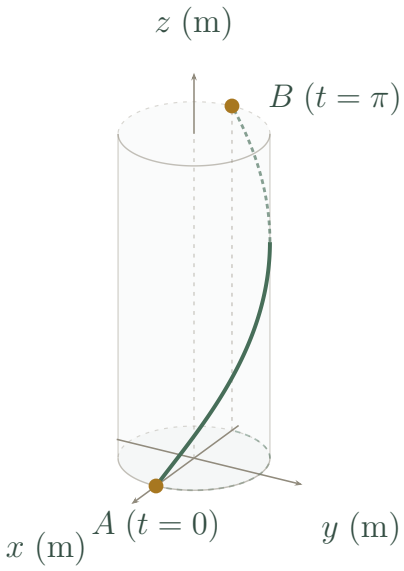
$L = \int_0^{\frac{\pi}{3}} \sec t \, dt$. [1 mark]

Using the supplied antiderivative, $L = [\ln|\sec t + \tan t|]_0^{\frac{\pi}{3}}$. [1 mark]

$L = \ln(2 + \sqrt{3}) - \ln 1 = \ln(2 + \sqrt{3})$. [1 mark]

Answer: $\ln(2 + \sqrt{3})$ units.

10. A cable follows $\mathbf{r}(t) = (2\cos t, 2\sin t, 3t)$ metres, $0 \le t \le \pi$. Its endpoints are A at $t = 0$ and B at $t = \pi$. [8 marks]



(a) Find the cable length and the straight-line distance between A and B. [5 marks]

$\mathbf{r}' = (-2\sin t, 2\cos t, 3)$ has magnitude $\sqrt{13}$. [1 mark]

The length is $\int_0^{\pi} \sqrt{13}\, dt = \pi\sqrt{13}$. [2 marks]

$A = (2, 0, 0)$ and $B = (-2, 0, 3\pi)$, giving displacement $(-4, 0, 3\pi)$. [1 mark]

The straight distance is $\sqrt{16 + 9\pi^2}$. [1 mark]

Answer: Cable length $\pi\sqrt{13}$ m; straight distance $\sqrt{16 + 9\pi^2}$ m.

(b) Prove algebraically that the cable length exceeds the endpoint separation. State why comparing squares is valid. [3 marks]

Both lengths are positive, so squaring preserves their order. [1 mark]

The difference of squares is $13\pi^2 - (16 + 9\pi^2) = 4\pi^2 - 16$. [1 mark]

Since $\pi > 2$, this is positive, proving the strict inequality. [1 mark]

Answer: The squared-length difference is $4\pi^2 - 16 > 0$, so the cable is longer.