Vector calculus - Test 5

Name: _____ Date: _____

10 questions | 66 marks | 90 minutes

Mixed applications, nonuniform parameterisations and optimisation. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

A scientific calculator may be used.

1. A moving point has position $\mathbf{r}(t) = (t^2, t^2 + 2)$ metres for $-2 \leq t \leq 3$ seconds. [6 marks]

(a) State the Cartesian path and describe the full tracing, including any reversal. [3 marks]

(b) Find its total distance and the magnitude of its displacement. [3 marks]

2. Two particles have positions $\mathbf{r}_A(t) = (t, t^2)$ and $\mathbf{r}_B(t) = (2 - t, t)$ metres for $t \geq 0$ seconds. [7 marks]

(a) Determine whether they collide, and if so give the time and position. [3 marks]

(b) Find the cosine of the angle between their velocity vectors at that instant. Do not round. [4 marks]

3. A particle has position $\mathbf{r}(t) = (e^t, e^{-t})$ metres for $t \geq 0$ seconds. [6 marks]

(a) Find the speed function. [3 marks]

(b) Prove that its minimum speed is attained at $t = 0$, and find the minimum. [3 marks]

4. A curve is $\mathbf{r}(t) = (t^3, t^2)$ for $-1 \le t \le 1$. [6 marks]

(a) Show why the usual derivative ratio is inconclusive at $t = 0$. [2 marks]

(b) Find the tangent line at the origin by considering the limiting direction of a secant from the origin. [4 marks]

5. A quadratic Bézier path is $\mathbf{B}(u) = (4u, 4u(1-u))$, $0 \le u \le 1$. A drawing head traverses it according to $u = t^2$ for $0 \le t \le 1$ seconds, with coordinates in metres. [7 marks]

(a) Find the position and velocity functions in terms of time t. [3 marks]

(b) The head is at rest initially even though $\mathbf{B}'(0) \neq \mathbf{0}$. Explain this, and find its speed at the greatest height. [4 marks]

6. A particle has velocity $\mathbf{v}(t) = (\cos t, \sin t, 1)$ m/s and initial position $(0, 0, 0)$ m. Consider $0 \le t \le \frac{\pi}{2}$ seconds. [6 marks]

(a) Find its position function and displacement over this interval. [4 marks]

(b) Find the distance travelled and average speed. [2 marks]

7. A smooth curve $\mathbf{r}(u)$, $a \le u \le b$, is reparameterised as $\mathbf{R}(t) = \mathbf{r}(\alpha t + \beta)$, where $\alpha > 0$. The t-interval is $\frac{a-\beta}{\alpha} \le t \le \frac{b-\beta}{\alpha}$. [7 marks]

(a) Find the relationship between the two derivative vectors and describe the orientation. [3 marks]

(b) Prove that the arc length is unchanged by this reparameterisation. [4 marks]

8. A spatial curve is $\mathbf{r}(t) = (t, \frac{t^2}{2}, \frac{t^3}{6})$, $0 \le t \le 2$. [5 marks]

(a) Show that its arc-length integrand is $1 + \frac{t^2}{2}$. [3 marks]

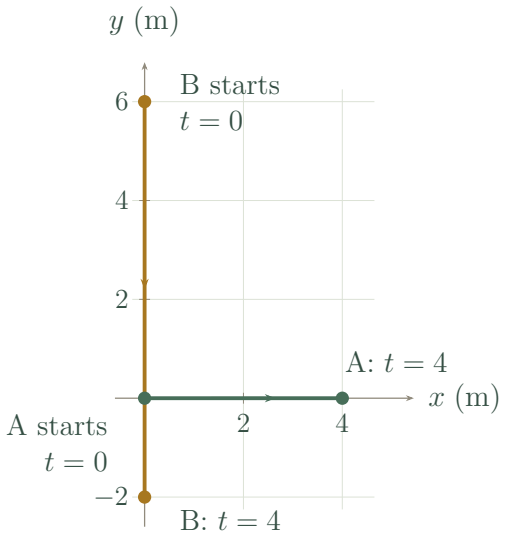
(b) Find the exact curve length. [2 marks]

9. A particle starts at rest and follows $\mathbf{r}(t) = (2\cos t^2, 2\sin t^2)$ metres for $0 \le t \le \sqrt{\pi}$ seconds. [7 marks]

(a) Find its initial acceleration and explain why rest does not imply zero acceleration. [4 marks]

(b) Find the total distance and the magnitude of the displacement. [3 marks]

10. Two robots move on perpendicular tracks. Their positions are $\mathbf{r}_A(t) = (t, 0)$ and $\mathbf{r}_B(t) = (0, 6 - 2t)$ metres for $0 \leq t \leq 4$ seconds. A safety rule requires their separation to remain at least 3 metres. [9 marks]



(a) Form an expression for the squared separation and find its minimum value on the interval. [4 marks]

(b) Determine whether the safety rule is satisfied, and find the full time interval on which it is violated. [5 marks]