Vector calculus - Test 5

10 questions | 66 marks | 90 minutes

Mixed applications, nonuniform parameterisations and optimisation. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

Equivalent correct methods earn the corresponding marks.

1. A moving point has position $\mathbf{r}(t) = (t^2, t^2 + 2)$ metres for $-2 \le t \le 3$ seconds. [6 marks]

(a) State the Cartesian path and describe the full tracing, including any reversal. [3 marks]

Eliminating t gives $y = x + 2$, with $0 \le x \le 9$. [1 mark]

For $-2 \le t \le 0$, it moves from $(4, 6)$ to $(0, 2)$. [1 mark]

It reverses at zero, then moves to $(9, 11)$, retracing the first segment before continuing. [1 mark]

Answer: $y = x + 2$, $0 \le x \le 9$; from $(4, 6)$ to $(0, 2)$, then to $(9, 11)$.

(b) Find its total distance and the magnitude of its displacement. [3 marks]

The two straight-stage lengths are $4\sqrt{2}$ and $9\sqrt{2}$, so total distance is $13\sqrt{2}$. [2 marks]

Endpoint displacement is $(5, 5)$, of magnitude $5\sqrt{2}$. [1 mark]

Answer: Distance $13\sqrt{2}$ m; displacement magnitude $5\sqrt{2}$ m.

2. Two particles have positions $\mathbf{r}_A(t) = (t, t^2)$ and $\mathbf{r}_B(t) = (2 - t, t)$ metres for $t \geq 0$ seconds. [7 marks]

(a) Determine whether they collide, and if so give the time and position. [3 marks]

Equality of x-coordinates gives $t = 2 - t$, so $t = 1$. [1 mark]

At this same time both y-coordinates are 1. [1 mark]

Therefore they collide at $(1, 1)$ m when $t = 1$ s. [1 mark]

Answer: They collide at $t = 1$ s at $(1, 1)$ m.

(b) Find the cosine of the angle between their velocity vectors at that instant. Do not round. [4 marks]

$\mathbf{v}_A(1) = (1, 2)$ and $\mathbf{v}_B(1) = (-1, 1)$. [1 mark]

The dot product is $-1 + 2 = 1$. [1 mark]

The magnitudes are $\sqrt{5}$ and $\sqrt{2}$. [1 mark]

Hence $\cos\theta = \frac{1}{\sqrt{5}\sqrt{2}} = \frac{1}{\sqrt{10}}$. [1 mark]

Answer: $\cos\theta = \frac{1}{\sqrt{10}}$.

3. A particle has position $\mathbf{r}(t) = (e^t, e^{-t})$ metres for $t \geq 0$ seconds. [6 marks]

(a) Find the speed function. [3 marks]

$\mathbf{v} = (e^t, -e^{-t})$. [2 marks]

Thus $s = \sqrt{e^{2t} + e^{-2t}}$. [1 mark]

Answer: $s(t) = \sqrt{e^{2t} + e^{-2t}}$ m/s.

(b) Prove that its minimum speed is attained at $t = 0$, and find the minimum. [3 marks]

$s'(t) = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$. [1 mark]

For $t \geq 0$, the numerator is nonnegative and is positive for $t > 0$, so speed increases after zero. [1 mark]

The minimum is $s(0) = \sqrt{2}$ m/s. [1 mark]

Answer: Minimum speed $\sqrt{2}$ m/s at $t = 0$.

4. A curve is $\mathbf{r}(t) = (t^3, t^2)$ for $-1 \le t \le 1$. [6 marks]

(a) Show why the usual derivative ratio is inconclusive at $t = 0$. [2 marks]

$\mathbf{r}'(t) = (3t^2, 2t)$, so both components vanish at zero. [1 mark]

Direct substitution into $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ therefore gives $\frac{0}{0}$, which does not determine a tangent. [1 mark]

Answer: $\mathbf{r}'(0) = \mathbf{0}$ and the ratio gives $\frac{0}{0}$.

(b) Find the tangent line at the origin by considering the limiting direction of a secant from the origin. [4 marks]

For $t \neq 0$, the secant vector is $(t^3, t^2) = t^2(t, 1)$. [1 mark]

A direction vector for this line is $(t, 1)$, which tends to $(0, 1)$ as $t \to 0$. [2 marks]

The limiting line through the origin is vertical: $x = 0$. [1 mark]

Answer: The tangent is $x = 0$.

5. A quadratic Bézier path is $\mathbf{B}(u) = (4u, 4u(1-u))$, $0 \le u \le 1$. A drawing head traverses it according to $u = t^2$ for $0 \le t \le 1$ seconds, with coordinates in metres. [7 marks]

(a) Find the position and velocity functions in terms of time t. [3 marks]

Substitute $u = t^2$ to obtain $\mathbf{r} = (4t^2, 4t^2 - 4t^4)$. [1 mark]

Differentiating with respect to time gives $\mathbf{v} = (8t, 8t - 16t^3)$. [2 marks]

Answer: $\mathbf{r}(t) = (4t^2, 4t^2 - 4t^4)$; $\mathbf{v}(t) = (8t, 8t - 16t^3)$.

(b) The head is at rest initially even though $\mathbf{B}'(0) \neq \mathbf{0}$. Explain this, and find its speed at the greatest height. [4 marks]

$\mathbf{v} = \mathbf{B}'(u)\frac{du}{dt}$, and $\frac{du}{dt} = 2t = 0$ initially. [1 mark]

The greatest height occurs at $u = \frac{1}{2}$, hence $t = \frac{1}{\sqrt{2}}$. [1 mark]

At that time $\mathbf{v} = (4\sqrt{2}, 0)$. [1 mark]

The speed is therefore $4\sqrt{2}$ m/s. [1 mark]

Answer: $\frac{du}{dt} = 0$ initially; at $t = \frac{1}{\sqrt{2}}$, speed is $4\sqrt{2}$ m/s.

6. A particle has velocity $\mathbf{v}(t) = (\cos t, \sin t, 1)$ m/s and initial position $(0, 0, 0)$ m. Consider $0 \le t \le \frac{\pi}{2}$ seconds. [6 marks]

(a) Find its position function and displacement over this interval. [4 marks]

Integrating the components gives $(\sin t + C_1,\ -\cos t + C_2, t + C_3)$. [1 mark]

The initial condition gives $C_1 = 0, C_2 = 1, C_3 = 0$. [1 mark]

Thus $\mathbf{r} = (\sin t, 1 - \cos t, t)$ and the endpoint displacement is $(1, 1, \frac{\pi}{2})$. [2 marks]

Answer: $\mathbf{r} = (\sin t, 1 - \cos t, t)$ m; displacement $(1, 1, \frac{\pi}{2})$ m.

(b) Find the distance travelled and average speed. [2 marks]

$|\mathbf{v}| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$, so distance is $\sqrt{2}(\frac{\pi}{2}) = \frac{\pi}{\sqrt{2}}$. [1 mark]

Dividing by elapsed time $\frac{\pi}{2}$ gives average speed $\sqrt{2}$. [1 mark]

Answer: Distance $\frac{\pi}{\sqrt{2}}$ m; average speed $\sqrt{2}$ m/s.

7. A smooth curve $\mathbf{r}(u)$, $a \le u \le b$, is reparameterised as $\mathbf{R}(t) = \mathbf{r}(\alpha t + \beta)$, where $\alpha > 0$. The t-interval is $\frac{a-\beta}{\alpha} \le t \le \frac{b-\beta}{\alpha}$. [7 marks]

(a) Find the relationship between the two derivative vectors and describe the orientation. [3 marks]

The chain rule gives $\mathbf{R}'(t) = \alpha \mathbf{r}'(\alpha t + \beta)$. [2 marks]

Because $\alpha > 0$, u increases with t, so the orientation is unchanged. [1 mark]

Answer: $\mathbf{R}'(t) = \alpha \mathbf{r}'(\alpha t + \beta)$; the orientation is unchanged.

(b) Prove that the arc length is unchanged by this reparameterisation. [4 marks]

Since $\alpha > 0$, $|\mathbf{R}'(t)| = \alpha |\mathbf{r}'(\alpha t + \beta)|$. [1 mark]

Let $u = \alpha t + \beta$, so $du = \alpha\, dt$. [1 mark]

The t-endpoints become $u = a, b$. [1 mark]

Substitution gives precisely $\int_a^b |\mathbf{r}'(u)|\, du$, the original length. [1 mark]

Answer: $\int_{\frac{a-\beta}{\alpha}}^{\frac{b-\beta}{\alpha}} |\mathbf{R}'(t)|\, dt = \int_a^b |\mathbf{r}'(u)|\, du$.

8. A spatial curve is $\mathbf{r}(t) = (t, \frac{t^2}{2}, \frac{t^3}{6})$, $0 \le t \le 2$. [5 marks]

(a) Show that its arc-length integrand is $1 + \frac{t^2}{2}$. [3 marks]

$\mathbf{r}' = (1, t, \frac{t^2}{2})$. [1 mark]

The squared magnitude is $1 + t^2 + \frac{t^4}{4} = (1 + \frac{t^2}{2})^2$. [1 mark]

Since $1 + \frac{t^2}{2} > 0$, the magnitude is $1 + \frac{t^2}{2}$. [1 mark]

Answer: $|\mathbf{r}'(t)| = 1 + \frac{t^2}{2}$.

(b) Find the exact curve length. [2 marks]

$L = \int_0^2 (1 + \frac{t^2}{2})\,dt$. [1 mark]

$L = [t + \frac{t^3}{6}]_0^2 = 2 + \frac{4}{3} = \frac{10}{3}$. [1 mark]

Answer: $\frac{10}{3}$ units.

9. A particle starts at rest and follows $\mathbf{r}(t) = (2\cos t^2, 2\sin t^2)$ metres for $0 \le t \le \sqrt{\pi}$ seconds. [7 marks]

(a) Find its initial acceleration and explain why rest does not imply zero acceleration. [4 marks]

$\mathbf{v} = (-4t\sin t^2, 4t\cos t^2)$. [1 mark]

$\mathbf{a} = (-4\sin t^2 - 8t^2\cos t^2, 4\cos t^2 - 8t^2\sin t^2)$. [1 mark]

At zero, $\mathbf{v} = \mathbf{0}$ but $\mathbf{a} = (0, 4)$. [1 mark]

Rest describes velocity at that instant; acceleration describes how velocity is changing. [1 mark]

Answer: $\mathbf{a}(0) = (0, 4)$ m/s^2; velocity can be zero at an instant while changing.

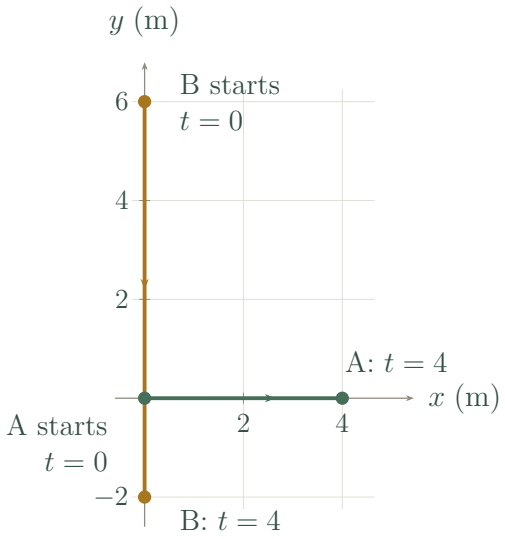
(b) Find the total distance and the magnitude of the displacement. [3 marks]

The speed is $4t$ for $t \ge 0$, so $D = \int_0^{\sqrt{\pi}} 4t\,dt = 2\pi$. [2 marks]

The endpoints are $(2, 0)$ and $(-2, 0)$, whose separation is 4 m. [1 mark]

Answer: Distance 2π m; displacement magnitude 4 m.

10. Two robots move on perpendicular tracks. Their positions are $\mathbf{r}_A(t) = (t, 0)$ and $\mathbf{r}_B(t) = (0, 6 - 2t)$ metres for $0 \le t \le 4$ seconds. A safety rule requires their separation to remain at least 3 metres. [9 marks]



(a) Form an expression for the squared separation and find its minimum value on the interval. [4 marks]

The relative position is $\mathbf{r}_A - \mathbf{r}_B = (t, 2t - 6)$. [1 mark]

$D^2 = t^2 + (2t - 6)^2 = 5t^2 - 24t + 36$. [1 mark]

Its derivative is $10t - 24$, zero at $t = \frac{12}{5}$ in the interval. [1 mark]

The upward-opening quadratic has minimum $\frac{36}{5}$. [1 mark]

Answer: $D^2 = 5t^2 - 24t + 36$ m^2; minimum $\frac{36}{5}$ m^2 at $t = \frac{12}{5}$ s.

(b) Determine whether the safety rule is satisfied, and find the full time interval on which it is violated. [5 marks]

The least separation is $\sqrt{\frac{36}{5}} = \frac{6}{\sqrt{5}} < 3$, so the rule is not satisfied. [1 mark]

Violation requires $D^2 < 9$, or $5t^2 - 24t + 27 < 0$. [1 mark]

The equality roots are $t = \frac{24 \pm \sqrt{576 - 540}}{10} = \frac{12 \pm 3}{5}$, namely $\frac{9}{5}$ and 3. [2 marks]

The quadratic is negative between its roots, so the rule is violated for $\frac{9}{5} < t < 3$ s. [1 mark]

Answer: It is violated for $\frac{9}{5} < t < 3$ seconds.