Vector calculus - Test 4

10 questions | 64 marks | 90 minutes

Spatial motion, circular motion and the geometry of acceleration. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

Equivalent correct methods earn the corresponding marks.

1. A survey drone moves in a straight line at constant velocity from $A = (1, -2, 3)$ m to $B = (7, 2, 3)$ m in 2 seconds, starting at $t = 0$. [5 marks]

(a) Give its position function for the journey. [3 marks]

$B - A = (6, 4, 0)$, so the constant velocity is $(3, 2, 0)$ m/s. [2 marks]

Thus $\mathbf{r}(t) = (1, -2, 3) + t(3, 2, 0)$ for $0 \le t \le 2$. [1 mark]

Answer: $\mathbf{r}(t) = (1, -2, 3) + t(3, 2, 0)$, $0 \le t \le 2$.

(b) Find the speed and explain what the zero third component of velocity means. [2 marks]

The speed is $\sqrt{3^2 + 2^2} = \sqrt{13}$ m/s. [1 mark]

The z-coordinate is constant at 3 m; zero vertical velocity does not mean rest. [1 mark]

Answer: Speed $\sqrt{13}$ m/s; height stays at $z = 3$ m.

2. A particle has velocity $\mathbf{v}(t) = (2t, 2, -1)$ m/s and initial position $(0, 1, 4)$ m. [7 marks]

(a) Find its position and acceleration functions. [4 marks]

Integrating gives $(t^2 + C_1, 2t + C_2, -t + C_3)$. [2 marks]

The initial position gives $\mathbf{r} = (t^2, 2t + 1, 4 - t)$. [1 mark]

Differentiating velocity gives $\mathbf{a} = (2, 0, 0)$. [1 mark]

Answer: $\mathbf{r} = (t^2, 2t + 1, 4 - t)$ m; $\mathbf{a} = (2, 0, 0)$ m/s^2.

(b) At $t = 1$, find its speed and a unit vector in its direction of motion. [3 marks]

$\mathbf{v}(1) = (2, 2, -1)$. [1 mark]

Its magnitude is $\sqrt{4 + 4 + 1} = 3$. [1 mark]

Divide by 3 to get $(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3})$. [1 mark]

Answer: Speed 3 m/s; unit direction $(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3})$.

3. A point on a rotating mechanism has position $\mathbf{r}(t) = (2\cos t, \sin t, 0)$ metres, where t is time in seconds. [6 marks]

(a) Describe its locus and find its velocities at $t = 0$ and $t = \frac{\pi}{2}$. [4 marks]

The locus is $\frac{x^2}{4} + y^2 = 1$ in the plane $z = 0$. [2 marks]

$\mathbf{v} = (-2\sin t, \cos t, 0)$, giving $(0, 1, 0)$ at zero and $(-2, 0, 0)$ at $\frac{\pi}{2}$. [2 marks]

Answer: An ellipse $\frac{x^2}{4} + y^2 = 1$ in $z = 0$; velocities $(0, 1, 0)$ and $(-2, 0, 0)$ m/s.

(b) Explain why a constant rate of change of the parameter does not give constant speed here. [2 marks]

The magnitudes of the two velocities are 1 and 2 m/s. [1 mark]

Constant phase rate on an ellipse need not give constant distance per unit time; the unequal coordinate scales yield different speeds. [1 mark]

Answer: The speeds at the two instants are 1 m/s and 2 m/s, so they are not constant.

4. A spatial curve is $\mathbf{r}(t) = (t^2, t, t^3)$. [5 marks]

(a) Find all parameter values at which the tangent direction is parallel to the plane $z = 0$. [3 marks]

$\mathbf{r}' = (2t, 1, 3t^2)$. [1 mark]

Parallelism to $z = 0$ requires third component $3t^2 = 0$, so $t = 0$. [1 mark]

The direction $(0, 1, 0)$ is nonzero, so a tangent is defined. [1 mark]

Answer: $t = 0$ only.

(b) Write the tangent line there. [2 marks]

The point is $\mathbf{r}(0) = (0, 0, 0)$. [1 mark]

Using direction $(0, 1, 0)$ gives the stated line. [1 mark]

Answer: $\mathbf{x} = (0, 0, 0) + \lambda(0, 1, 0)$, $\lambda \in \mathbb{R}$.

5. A particle moves uniformly on a circle according to $\mathbf{r}(t) = \mathbf{c} + (R\cos\omega t, R\sin\omega t)$, where $\mathbf{c}$ is fixed, $R > 0$ and $\omega \neq 0$ are constants. [7 marks]

(a) Prove that $\mathbf{a} = -\omega^2(\mathbf{r} - \mathbf{c})$ and interpret the direction of acceleration. [4 marks]

$\mathbf{v} = (-R\omega\sin\omega t, R\omega\cos\omega t)$. [1 mark]

$\mathbf{a} = (-R\omega^2\cos\omega t,\ -R\omega^2\sin\omega t)$. [1 mark]

This equals $-\omega^2(\mathbf{r} - \mathbf{c})$. [1 mark]

Since $\omega^2 > 0$, acceleration is opposite the outward radius vector and points towards the centre. [1 mark]

Answer: $\mathbf{a} = -\omega^2(\mathbf{r} - \mathbf{c})$; acceleration points towards the centre.

(b) Express its acceleration magnitude in terms of its speed s and radius R. [3 marks]

$s = R|\omega|$. [1 mark]

$|\mathbf{a}| = R\omega^2$. [1 mark]

Eliminating ω gives $|\mathbf{a}| = \frac{s^2}{R}$. [1 mark]

Answer: $|\mathbf{a}| = \frac{s^2}{R}$.

6. A helical path has position $\mathbf{r}(t) = (2\cos 3t, 2\sin 3t, t)$ metres, $t \geq 0$ seconds. [6 marks]

(a) Find the time for one turn, the rise per turn and the speed. [4 marks]

A turn requires $3T = 2\pi$, hence $T = \frac{2\pi}{3}$ s. [1 mark]

The height increases by $T = \frac{2\pi}{3}$ m. [1 mark]

$\mathbf{v} = (-6\sin 3t, 6\cos 3t, 1)$, so the speed is $\sqrt{36+1} = \sqrt{37}$. [2 marks]

Answer: Time $\frac{2\pi}{3}$ s; rise $\frac{2\pi}{3}$ m; speed $\sqrt{37}$ m/s.

(b) Does the complete position function have a positive period? Explain. [2 marks]

Although x and y repeat after $\frac{2\pi}{3}$, $z(t+T) = z(t) + T$. [1 mark]

For every $T > 0$, the third coordinates differ, so the full position never repeats periodically. [1 mark]

Answer: No: the height increases by T for every positive time shift T.

7. A particle has position $\mathbf{r}(t) = (t^2, t^2 - 2t, 2t)$ metres for $0 \leq t \leq 2$ seconds. [6 marks]

(a) Find its speed squared as a completed square. [3 marks]

$\mathbf{v} = (2t, 2t-2, 2)$. [1 mark]

$s^2 = 4t^2 + (2t-2)^2 + 4 = 8t^2 - 8t + 8$. [1 mark]

Complete the square: $s^2 = 8(t - \frac{1}{2})^2 + 6$. [1 mark]

Answer: $s^2 = 8(t - \frac{1}{2})^2 + 6$.

(b) Find the minimum speed and the acceleration when it occurs. Is minimum speed the same as zero acceleration? [3 marks]

The square is least at $t = \frac{1}{2}$, giving speed $\sqrt{6}$. [1 mark]

$\mathbf{a} = (2, 2, 0)$ at every time. [1 mark]

Thus minimum speed can occur with nonzero acceleration, which changes the direction of velocity. [1 mark]

Answer: Minimum speed $\sqrt{6}$ m/s at $t = \frac{1}{2}$ s; acceleration $(2, 2, 0)$ m/s^2, which is nonzero.

8. A curve has position $\mathbf{r}(t) = (\sin t\cos t, \sin^2 t, \cos t)$. [6 marks]

(a) Prove that the curve lies on the unit sphere centred at the origin. [3 marks]

$|\mathbf{r}|^2 = \sin^2 t\cos^2 t + \sin^4 t + \cos^2 t$. [1 mark]

The first two terms combine to $\sin^2 t(\cos^2 t + \sin^2 t) = \sin^2 t$. [1 mark]

Hence $|\mathbf{r}|^2 = \sin^2 t + \cos^2 t = 1$. [1 mark]

Answer: $|\mathbf{r}(t)|^2 = 1$.

(b) At any point where the derivative is nonzero, prove that the tangent direction is perpendicular to the position vector. [3 marks]

Differentiate $\mathbf{r} \cdot \mathbf{r} = 1$. [1 mark]

The componentwise product rule gives $2\mathbf{r} \cdot \mathbf{r}' = 0$. [1 mark]

Since the position vector has length 1 and the tangent direction is assumed nonzero, their zero dot product proves perpendicularity. [1 mark]

Answer: $\mathbf{r} \cdot \mathbf{r}' = 0$, so the nonzero tangent direction is perpendicular to the radius.

9. A warehouse robot starts at $(0, 0, 0)$ m. For $0 \le t \le 2$ seconds its velocity is $(3, 0, 0)$ m/s. For $2 < t \le 5$ seconds its velocity is $(0, 0, 4)$ m/s. Treat the instantaneous change at $t = 2$ as an idealisation. [6 marks]

(a) Find its final position and total distance travelled. [3 marks]

The stage displacements are $2(3, 0, 0) = (6, 0, 0)$ and $3(0, 0, 4) = (0, 0, 12)$. [1 mark]

The final position is $(6, 0, 12)$ m. [1 mark]

The total distance is $2(3) + 3(4) = 18$ m. [1 mark]

Answer: Final position $(6, 0, 12)$ m; distance 18 m.

(b) Compare the magnitude of average velocity with average speed over the full 5 seconds. [3 marks]

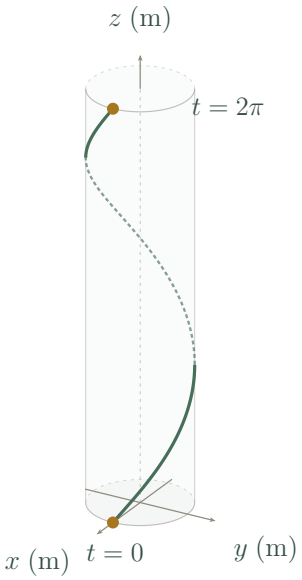
Average velocity is $(\frac{6}{5}, 0, \frac{12}{5})$ m/s. [1 mark]

Its magnitude is $\frac{\sqrt{36 + 144}}{5} = \frac{6\sqrt{5}}{5}$ m/s. [1 mark]

Average speed is $\frac{18}{5}$ m/s, greater because $18 > 6\sqrt{5}$. [1 mark]

Answer: Average-velocity magnitude $\frac{6\sqrt{5}}{5}$ m/s; average speed $\frac{18}{5}$ m/s, which is greater.

10. A drone follows the helix $\mathbf{r}(t) = (3\cos t, 3\sin t, 4t)$ metres, $0 \le t \le 2\pi$, where t is in seconds. [10 marks]



(a) Find its velocity and speed, and hence the distance travelled during the ascent. [4 marks]

$\mathbf{v} = (-3\sin t, 3\cos t, 4)$. [1 mark]

$s = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$ m/s. [1 mark]

$D = \int_0^{2\pi} 5\,dt = 10\pi$ m. [2 marks]

Answer: $\mathbf{v} = (-3\sin t, 3\cos t, 4)$ m/s; speed 5 m/s; distance 10π m.

(b) Find the endpoint separation, then the angle between the velocity and the horizontal plane. Give the angle to the nearest degree. [4 marks]

The endpoints are $(3, 0, 0)$ and $(3, 0, 8\pi)$, so the separation is 8π m. [2 marks]

The horizontal velocity has magnitude 3 and the vertical component is 4, so $\tan\theta = \frac{4}{3}$. [1 mark]

$\theta \approx 53.13^\circ$, hence 53°. [1 mark]

Answer: Separation 8π m; angle 53°.

(c) Explain why the drone has nonzero acceleration even though its speed is constant. [2 marks]

Differentiating velocity gives $\mathbf{a} = (-3\cos t,\ -3\sin t, 0) \neq \mathbf{0}$. [1 mark]

Constant speed fixes the magnitude of velocity; its horizontal direction changes around the cylinder. [1 mark]

Answer: $\mathbf{a} = (-3\cos t,\ -3\sin t, 0)$ m/s^2 changes the horizontal direction of velocity.