Vector calculus - Test 3

Name: _____ Date: _____

10 questions | 63 marks | 90 minutes

Polynomial curves and Bézier design, followed by connected modelling tasks. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

A scientific calculator may be used.

1. A pen must draw the straight segment from $A = (1, -2)$ to $B = (7, 1)$, starting at A when $u = 0$ and finishing at B when $u = 1$. [5 marks]

(a) Give a linear vector parameterisation and state its interval. [3 marks]

(b) A second pen uses the same rule for $0 \leq u \leq 2$. Explain the change to the traced segment. [2 marks]

2. A point follows $x = t^3 - 3t$, $y = t^2$, for $-2 \le t \le 2$. [7 marks]

(a) Find the parameter values where x has a stationary value, and classify the two stationary values of x. [4 marks]

(b) Find the tangent directions there and determine whether the point stops. [3 marks]

3. A quadratic Bézier curve has controls $P_0 = (0, 0)$, $P_1 = (1, 3)$, $P_2 = (4, 2)$ and rule
$\mathbf{B}(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$
where $0 \le u \le 1$. [6 marks]

(a) Expand its two coordinate functions. [2 marks]

(b) Find the greatest y-coordinate on the curve and the corresponding point. [4 marks]

4. A cubic Bézier curve has endpoints $P_0 = (0, 0)$, $P_3 = (6, 0)$. Its required endpoint derivatives are $\mathbf{B}'(0) = (3, 6)$ and $\mathbf{B}'(1) = (3, -3)$. You may use $\mathbf{B}'(0) = 3(P_1 - P_0)$ and $\mathbf{B}'(1) = 3(P_3 - P_2)$. [6 marks]

(a) Determine both interior control points. [4 marks]

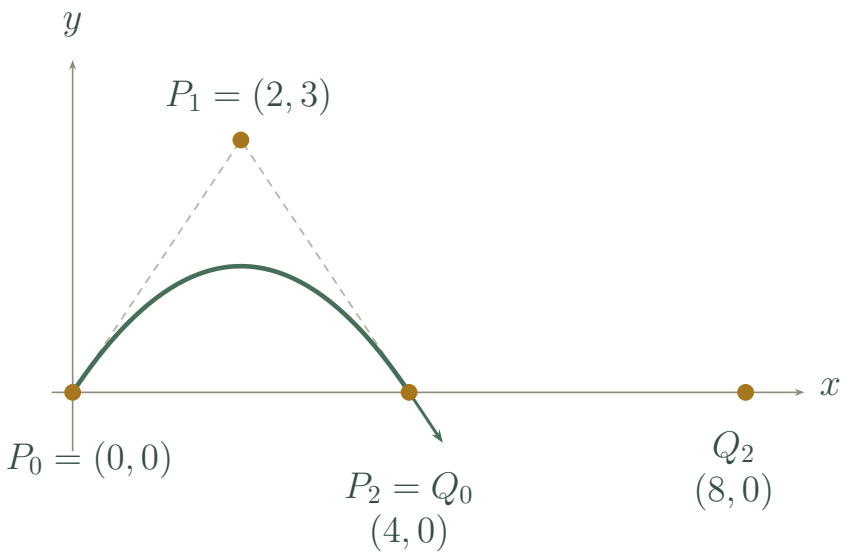
(b) Using the cubic Bézier weights, find $\mathbf{B}(\frac{1}{2})$. [2 marks]

5. For arbitrary vector control points P_0, P_1, P_2, let $\mathbf{B}(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$. [5 marks]

(a) Prove that $\mathbf{B}(\frac{1}{2}) = \frac{P_0 + 2P_1 + P_2}{4}$. [2 marks]

(b) Prove that $\mathbf{B}(\frac{1}{2}) = P_1$ if and only if P_1 is the midpoint of the two endpoint controls. [3 marks]

6. Two quadratic Bézier paths are to be joined. The first has controls $P_0 = (0, 0)$, $P_1 = (2, 3)$, $P_2 = (4, 0)$. The second has controls $Q_0 = P_2$, unknown Q_1, and $Q_2 = (8, 0)$. Each uses its standard parameter interval from 0 to 1. [8 marks]



(a) Find the final derivative of the first path. [2 marks]

(b) Find Q_1 so that the derivatives at the join are equal. [3 marks]

(c) A designer instead chooses $Q_1 = (5, -\frac{3}{2})$. Explain whether the two curves still have the same tangent direction, and whether a pen using each standard parameter as time would have continuous velocity at the join. [3 marks]

7. A spatial curve is $\mathbf{r}(t) = (t, t^2, t^3)$. [5 marks]

(a) Find the point and tangent direction at $t = 1$. [2 marks]

(b) Write a vector equation of the tangent line using a new parameter λ, and find where that line has $z = 3$. [3 marks]

8. A three-dimensional animation uses acceleration $\mathbf{a}(t) = (6t, 2, 0)$, initial velocity $(0, -1, 2)$ and initial position $(1, 3, -2)$. Time is in seconds and distances in metres. [7 marks]

(a) Determine its position function. [4 marks]

(b) Find the displacement from $t = 0$ to $t = 1$ and show that the point cannot be at rest at any time. [3 marks]

9. A polynomial curve is $\mathbf{r}(t) = (t - \frac{t^3}{3}, t^2)$, $0 \le t \le 1$. [6 marks]

(a) Show that its speed with respect to the parameter is $1 + t^2$. [3 marks]

(b) Find its exact length and compare it with the endpoint separation. [3 marks]

10. A camera path is designed as $\mathbf{r}(t) = (at, bt - t^2)$ metres for $0 \le t \le 4$ seconds, with $a > 0$. It must start at the origin and finish at $(12, 0)$ m. Its permitted speed is at most 5 m/s. [8 marks]

(a) Find a and b, and find the greatest height. [4 marks]

(b) Prove that the speed limit is satisfied throughout the journey. State when the limit is reached. [4 marks]