Vector calculus - Test 3

10 questions | 63 marks | 90 minutes

Polynomial curves and Bézier design, followed by connected modelling tasks. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

Equivalent correct methods earn the corresponding marks.

1. A pen must draw the straight segment from $A = (1, -2)$ to $B = (7, 1)$, starting at A when $u = 0$ and finishing at B when $u = 1$. [5 marks]

(a) Give a linear vector parameterisation and state its interval. [3 marks]

$B - A = (6, 3)$. [1 mark]

$\mathbf{r}(u) = (1, -2) + u(6, 3)$. [1 mark]

The interval is $0 \le u \le 1$. [1 mark]

Answer: $\mathbf{r}(u) = (1, -2) + u(6, 3),\ 0 \le u \le 1$.

(b) A second pen uses the same rule for $0 \le u \le 2$. Explain the change to the traced segment. [2 marks]

$\mathbf{r}(2) = (13, 4)$. [1 mark]

The constant nonzero derivative $(6, 3)$ means it continues in the same direction beyond B. [1 mark]

Answer: It continues past B to $(13, 4)$; it does not retrace the original segment.

2. A point follows $x = t^3 - 3t$, $y = t^2$, for $-2 \le t \le 2$. [7 marks]

(a) Find the parameter values where x has a stationary value, and classify the two stationary values of x. [4 marks]

$\frac{dx}{dt} = 3t^2 - 3 = 0$ gives $t = \pm 1$. [2 marks]

$\frac{d^2x}{dt^2} = 6t$ is negative at -1 and positive at 1. [1 mark]

$x(-1) = 2$ and $x(1) = -2$. [1 mark]

Answer: At $t = -1$, x has a local maximum of 2; at $t = 1$, x has a local minimum of -2.

(b) Find the tangent directions there and determine whether the point stops. [3 marks]

$\mathbf{r}' = (3t^2 - 3, 2t)$. [1 mark]

At $t = -1, 1$ this gives $(0, -2), (0, 2)$. [1 mark]

Both vectors are nonzero and vertical, so neither coordinate turning point is a stop. [1 mark]

Answer: Vertical directions $(0, -2)$ and $(0, 2)$; it does not stop at either point.

3. A quadratic Bézier curve has controls $P_0 = (0, 0)$, $P_1 = (1, 3)$, $P_2 = (4, 2)$ and rule
$\mathbf{B}(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$
where $0 \le u \le 1$. [6 marks]

(a) Expand its two coordinate functions. [2 marks]

$x = 2u(1-u) + 4u^2 = 2u + 2u^2$. [1 mark]

$y = 6u(1-u) + 2u^2 = 6u - 4u^2$. [1 mark]

Answer: $x = 2u + 2u^2$, $y = 6u - 4u^2$.

(b) Find the greatest y-coordinate on the curve and the corresponding point. [4 marks]

$\frac{dy}{du} = 6 - 8u = 0$ gives $u = \frac{3}{4}$. [1 mark]

The second derivative is $-8 < 0$, and $y(0) = 0$, $y(1) = 2$, so this is the greatest y-value. [1 mark]

$y(\frac{3}{4}) = \frac{9}{4}$. [1 mark]

$x(\frac{3}{4}) = \frac{21}{8}$, giving $(\frac{21}{8}, \frac{9}{4})$. [1 mark]

Answer: Maximum $y = \frac{9}{4}$ at $u = \frac{3}{4}$; point $(\frac{21}{8}, \frac{9}{4})$.

4. A cubic Bézier curve has endpoints $P_0 = (0, 0)$, $P_3 = (6, 0)$. Its required endpoint derivatives are $\mathbf{B}'(0) = (3, 6)$ and $\mathbf{B}'(1) = (3, -3)$. You may use $\mathbf{B}'(0) = 3(P_1 - P_0)$ and $\mathbf{B}'(1) = 3(P_3 - P_2)$. [6 marks]

(a) Determine both interior control points. [4 marks]

$P_1 = P_0 + \frac{1}{3}(3, 6) = (1, 2)$. [2 marks]

$P_2 = P_3 - \frac{1}{3}(3, -3) = (5, 1)$. [2 marks]

Answer: $P_1 = (1, 2)$ and $P_2 = (5, 1)$.

(b) Using the cubic Bézier weights, find $\mathbf{B}(\frac{1}{2})$. [2 marks]

$\mathbf{B}(\frac{1}{2}) = \frac{1}{8}P_0 + \frac{3}{8}P_1 + \frac{3}{8}P_2 + \frac{1}{8}P_3$. [1 mark]

Substitution gives $(3, \frac{9}{8})$. [1 mark]

Answer: $\mathbf{B}(\frac{1}{2}) = (3, \frac{9}{8})$.

5. For arbitrary vector control points P_0, P_1, P_2, let $\mathbf{B}(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$. [5 marks]

(a) Prove that $\mathbf{B}(\frac{1}{2}) = \frac{P_0 + 2P_1 + P_2}{4}$. [2 marks]

At $u = \frac{1}{2}$, the scalar weights are $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. [1 mark]

Combining over denominator 4 gives the stated identity. [1 mark]

Answer: $\mathbf{B}(\frac{1}{2}) = \frac{P_0 + 2P_1 + P_2}{4}$.

(b) Prove that $\mathbf{B}(\frac{1}{2}) = P_1$ if and only if P_1 is the midpoint of the two endpoint controls. [3 marks]

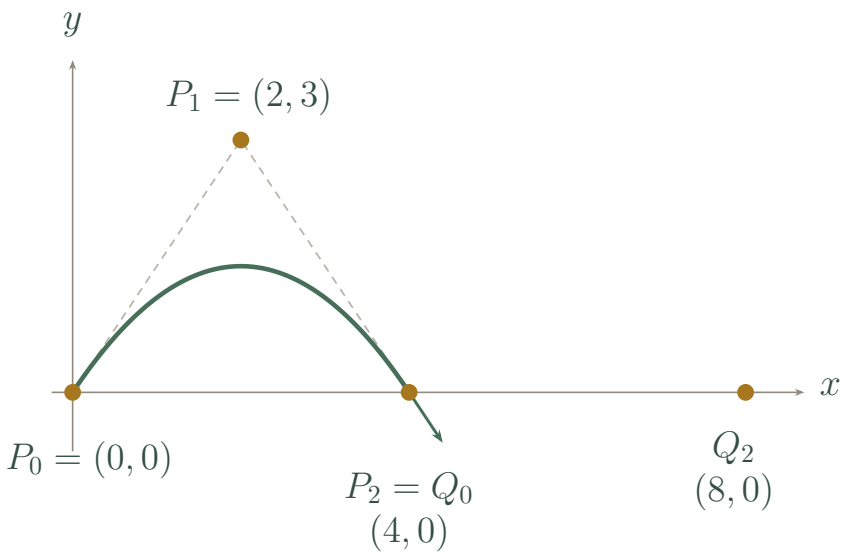
$\mathbf{B}(\frac{1}{2}) = P_1 \iff P_0 + 2P_1 + P_2 = 4P_1$. [1 mark]

Rearranging gives $P_0 + P_2 = 2P_1$, so $P_1 = \frac{P_0 + P_2}{2}$. [1 mark]

Each algebraic step is reversible, proving both directions. [1 mark]

Answer: $\mathbf{B}(\frac{1}{2}) = P_1 \iff P_1 = \frac{P_0 + P_2}{2}$.

6. Two quadratic Bézier paths are to be joined. The first has controls $P_0 = (0, 0)$, $P_1 = (2, 3)$, $P_2 = (4, 0)$. The second has controls $Q_0 = P_2$, unknown Q_1, and $Q_2 = (8, 0)$. Each uses its standard parameter interval from 0 to 1. [8 marks]



(a) Find the final derivative of the first path. [2 marks]

For a quadratic curve, $\mathbf{B}'(1) = 2(P_2 - P_1)$. [1 mark]

This gives $2(2, -3) = (4, -6)$. [1 mark]

Answer: $\mathbf{B}'(1) = (4, -6)$.

(b) Find Q_1 so that the derivatives at the join are equal. [3 marks]

The second initial derivative is $2(Q_1 - Q_0)$. [1 mark]

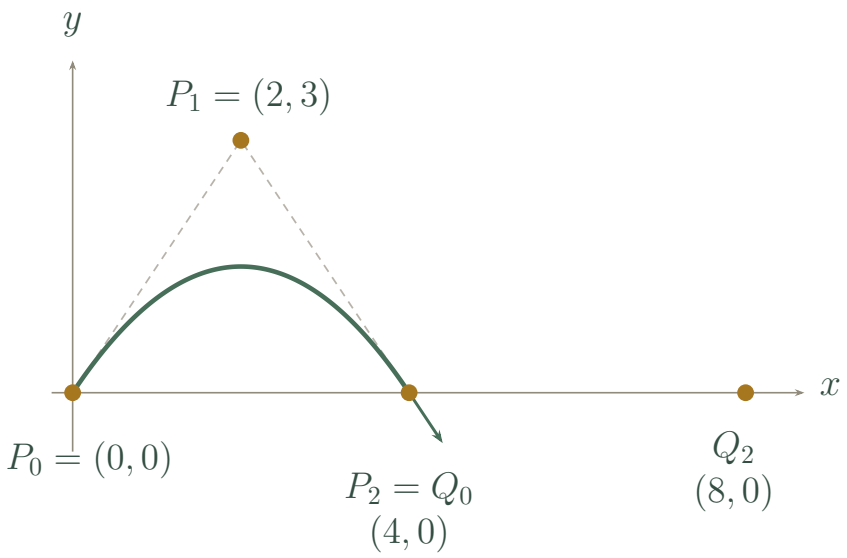
Set $2(Q_1 - (4, 0)) = (4, -6)$. [1 mark]

Then $Q_1 = (6, -3)$. [1 mark]

Answer: $Q_1 = (6, -3)$.

Question 6 (continued)

Two quadratic Bézier paths are to be joined. The first has controls $P_0 = (0, 0)$, $P_1 = (2, 3)$, $P_2 = (4, 0)$. The second has controls $Q_0 = P_2$, unknown Q_1, and $Q_2 = (8, 0)$. Each uses its standard parameter interval from 0 to 1.



(c) A designer instead chooses $Q_1 = (5, -\frac{3}{2})$. Explain whether the two curves still have the same tangent direction, and whether a pen using each standard parameter as time would have continuous velocity at the join. [3 marks]

The initial first-derivative vector of the second path is $2((5, -\frac{3}{2}) - (4, 0)) = (2, -3)$. [1 mark]

Since $(2, -3) = \frac{1}{2}(4, -6)$, the vectors have the same forward direction. [1 mark]

Their magnitudes differ. If each standard parameter is time, both speed and velocity change at the join, so velocity is not continuous. [1 mark]

Answer: Same forward tangent direction; velocity is not continuous, because the initial first-derivative vector of the second path is $\frac{1}{2}$ of the final first-derivative vector of the first path.

7. A spatial curve is $\mathbf{r}(t) = (t, t^2, t^3)$. [5 marks]

(a) Find the point and tangent direction at $t = 1$. [2 marks]

$\mathbf{r}(1) = (1, 1, 1)$. [1 mark]

$\mathbf{r}'(t) = (1, 2t, 3t^2)$ gives $(1, 2, 3)$. [1 mark]

Answer: Point $(1, 1, 1)$; direction $(1, 2, 3)$.

(b) Write a vector equation of the tangent line using a new parameter λ, and find where that line has $z = 3$. [3 marks]

The tangent is $\mathbf{x} = (1, 1, 1) + \lambda(1, 2, 3)$, $\lambda \in \mathbb{R}$. [1 mark]

$1 + 3\lambda = 3$ gives $\lambda = \frac{2}{3}$. [1 mark]

The line then contains $(\frac{5}{3}, \frac{7}{3}, 3)$. [1 mark]

Answer: $\mathbf{x} = (1, 1, 1) + \lambda(1, 2, 3)$; point $(\frac{5}{3}, \frac{7}{3}, 3)$.

8. A three-dimensional animation uses acceleration $\mathbf{a}(t) = (6t, 2, 0)$, initial velocity $(0, -1, 2)$ and initial position $(1, 3, -2)$. Time is in seconds and distances in metres. [7 marks]

(a) Determine its position function. [4 marks]

Integrating and using velocity at zero gives $\mathbf{v} = (3t^2, 2t - 1, 2)$. [2 marks]

Integrating and using position at zero gives $\mathbf{r} = (t^3 + 1, t^2 - t + 3, 2t - 2)$. [2 marks]

Answer: $\mathbf{r}(t) = (t^3 + 1, t^2 - t + 3, 2t - 2)$ m.

(b) Find the displacement from $t = 0$ to $t = 1$ and show that the point cannot be at rest at any time. [3 marks]

$\mathbf{r}(1) = (2, 3, 0)$, so $\mathbf{r}(1) - \mathbf{r}(0) = (1, 0, 2)$. [2 marks]

The velocity has z-component 2 at every time, so it cannot be the zero vector. [1 mark]

Answer: Displacement $(1, 0, 2)$ m; never at rest because the z-component of velocity is 2.

9. A polynomial curve is $\mathbf{r}(t) = (t - \frac{t^3}{3}, t^2)$, $0 \le t \le 1$. [6 marks]

(a) Show that its speed with respect to the parameter is $1 + t^2$. [3 marks]

$\mathbf{r}' = (1 - t^2, 2t)$. [1 mark]

$|\mathbf{r}'|^2 = (1 - t^2)^2 + 4t^2 = (1 + t^2)^2$. [1 mark]

Since $1 + t^2 > 0$, the magnitude is $1 + t^2$. [1 mark]

Answer: $|\mathbf{r}'(t)| = 1 + t^2$.

(b) Find its exact length and compare it with the endpoint separation. [3 marks]

$L = \int_0^1 (1 + t^2)\,dt = \frac{4}{3}$. [1 mark]

The endpoint displacement is $(\frac{2}{3}, 1)$, of magnitude $\frac{\sqrt{13}}{3}$. [1 mark]

Since $16 > 13$, $\frac{4}{3} > \frac{\sqrt{13}}{3}$. [1 mark]

Answer: Length $\frac{4}{3}$; separation $\frac{\sqrt{13}}{3}$; the length is greater.

10. A camera path is designed as $\mathbf{r}(t) = (at, bt - t^2)$ metres for $0 \le t \le 4$ seconds, with $a > 0$. It must start at the origin and finish at $(12, 0)$ m. Its permitted speed is at most 5 m/s. [8 marks]

(a) Find a and b, and find the greatest height. [4 marks]

At $t = 4$, $4a = 12$, so $a = 3$. [1 mark]

Also $4b - 16 = 0$, so $b = 4$. [1 mark]

$y = 4t - t^2 = 4 - (t - 2)^2$, giving maximum height 4 m at $t = 2$. [2 marks]

Answer: $a = 3$, $b = 4$; greatest height 4 m at $t = 2$ s.

(b) Prove that the speed limit is satisfied throughout the journey. State when the limit is reached. [4 marks]

$\mathbf{v} = (3, 4 - 2t)$, so $s^2 = 9 + (4 - 2t)^2$. [1 mark]

For $0 \le t \le 4$, $-4 \le 4 - 2t \le 4$, so $(4 - 2t)^2 \le 16$. [1 mark]

Therefore $s^2 \le 25$ and $s \le 5$ m/s. [1 mark]

Equality holds at $t = 0, 4$ only. [1 mark]

Answer: The speed is at most 5 m/s, with equality at $t = 0$ and $t = 4$.