Vector calculus - Test 2

Name: _____ Date: _____

10 questions | 62 marks | 90 minutes

Rates, tangent directions, collisions and mathematical justification. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

A scientific calculator may be used.

1. A curve has $x = 4\cos t$, $y = 2\sin t$, $0 \leq t \leq \pi$. [5 marks]

(a) Give its Cartesian equation and specify the portion traced. [3 marks]

(b) Describe the orientation, including the endpoints. [2 marks]

2. Two automated carts use the same coordinate grid in metres and the same clock in seconds. Cart A has position $(t, 2t)$ and cart B has position $(4 - t, 1 + t)$ for $t \geq 0$. [6 marks]

(a) Find where the two geometric paths intersect. You may use different parameters for the two carts. [3 marks]

(b) Determine whether the carts collide, explaining the role of time. [3 marks]

3. The curve $\mathbf{r}(t) = (t^2, t^3 - t)$ is traced for $-1 \leq t \leq 1$. [6 marks]

(a) Show that the two endpoints of the parameter interval give the same point. [2 marks]

(b) Find the two tangent lines at this repeated point and explain why they are different. [4 marks]

4. A particle has acceleration $\mathbf{a}(t) = (6t, -2)$ m/s^2, velocity $\mathbf{v}(0) = (1, 4)$ m/s and position $\mathbf{r}(0) = (2, -1)$ m. [7 marks]

(a) Find the velocity and position functions. [4 marks]

(b) For $t \geq 0$, find the position and velocity at the greatest height. [3 marks]

5. A curve is given by $x = t^2$, $y = t$. [5 marks]

(a) At $t = 0$, find the derivative vector and use it to determine the tangent line. [3 marks]

(b) Explain why an undefined value of $\frac{dy}{dx}$ at this point does not mean that the particle is at rest. [2 marks]

6. A particle moves on a unit circle according to $\mathbf{r}(t) = (\cos(t^2), \sin(t^2))$ for $1 \le t \le 2$ seconds; positions are in metres. [6 marks]

(a) Find the velocity and speed functions. [3 marks]

(b) Find the distance travelled from $t = 1$ to $t = 2$, and the rate of change of speed. [3 marks]

7. A cubic Bézier curve is
$\mathbf{B}(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u)P_2 + u^3 P_3$
where $0 \le u \le 1$ and $P_0 \ne P_1$, $P_2 \ne P_3$. [6 marks]

(a) Prove that $\mathbf{B}'(0) = 3(P_1 - P_0)$. [3 marks]

(b) By evaluating the same derivative at $u = 1$, establish the final tangent direction. [3 marks]

8. A cleaning robot follows $\mathbf{r}(t) = (3\sin t, 3\cos t)$ metres, $0 \le t \le 3\pi$, with time in seconds. [6 marks]

(a) Find the total distance travelled. [3 marks]

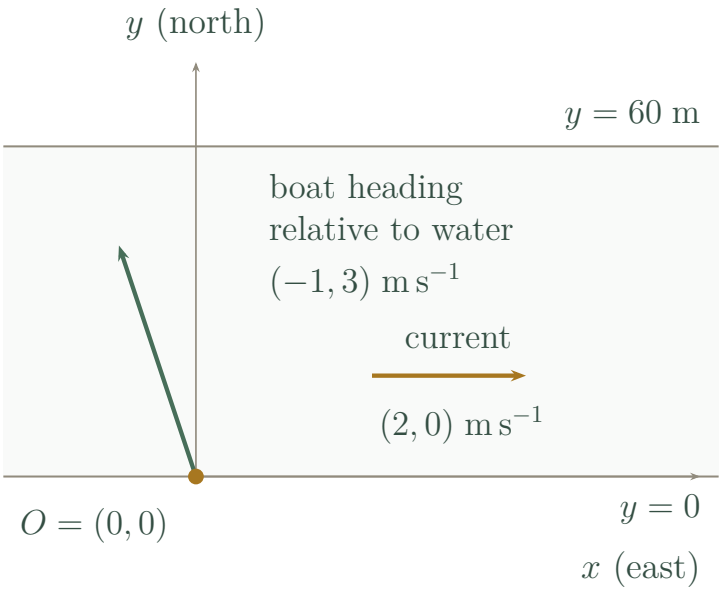
(b) Find the displacement and explain why its magnitude differs from the distance. [3 marks]

9. Let $\mathbf{r}(t)$ be twice differentiable on an interval, with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. [6 marks]

(a) Prove that constant speed implies $\mathbf{v} \cdot \mathbf{a} = 0$. [3 marks]

(b) If $\mathbf{v} \cdot \mathbf{a} = 0$ at only one instant, must speed be constant throughout the interval? Justify with a counterexample. [3 marks]

10. A straight river has banks $y=0$ and $y=60$ metres, with x measured east and y north. A boat starts at the origin. The current is $(2,0)$ m/s. Initially the boat maintains velocity $(-1,3)$ m/s relative to the water. [9 marks]



(a) Find its velocity relative to the ground and where it first reaches the opposite bank. [4 marks]

(b) The boat can instead choose a different constant heading, while keeping its speed relative to the water equal to $\sqrt{10}$ m/s. Find the heading vector and crossing time needed to arrive directly opposite its start. [5 marks]