Vector calculus - Test 2

10 questions | 62 marks | 90 minutes

Rates, tangent directions, collisions and mathematical justification. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

Equivalent correct methods earn the corresponding marks.

1. A curve has $x = 4\cos t$, $y = 2\sin t$, $0 \le t \le \pi$. [5 marks]

(a) Give its Cartesian equation and specify the portion traced. [3 marks]

Using $\cos^2 t + \sin^2 t = 1$ gives $\frac{x^2}{16} + \frac{y^2}{4} = 1$. [2 marks]

Since $0 \le t \le \pi$, the upper half, $y \ge 0$, is traced. [1 mark]

Answer: $\frac{x^2}{16} + \frac{y^2}{4} = 1$, with $y \ge 0$.

(b) Describe the orientation, including the endpoints. [2 marks]

$t = 0, \frac{\pi}{2}, \pi$ give $(4, 0), (0, 2), (-4, 0)$. [1 mark]

Increasing t traces the upper half once anticlockwise. [1 mark]

Answer: From $(4, 0)$, through $(0, 2)$, to $(-4, 0)$, once anticlockwise.

2. Two automated carts use the same coordinate grid in metres and the same clock in seconds. Cart A has position $(t, 2t)$ and cart B has position $(4-t, 1+t)$ for $t \geq 0$. [6 marks]

(a) Find where the two geometric paths intersect. You may use different parameters for the two carts. [3 marks]

The path equations are $y = 2x$ for A and $y = 5 - x$ for B. [1 mark]

$2x = 5 - x$ gives $x = \frac{5}{3}$, $y = \frac{10}{3}$. [2 marks]

Answer: The paths intersect at $(\frac{5}{3}, \frac{10}{3})$ m.

(b) Determine whether the carts collide, explaining the role of time. [3 marks]

A reaches the intersection at $t = \frac{5}{3}$ s. [1 mark]

For B, $4 - t = \frac{5}{3}$ gives $t = \frac{7}{3}$ s. [1 mark]

The intersection is reached at different times, so there is no collision. [1 mark]

Answer: They do not collide: A reaches the intersection at $\frac{5}{3}$ s, B at $\frac{7}{3}$ s.

3. The curve $\mathbf{r}(t) = (t^2, t^3 - t)$ is traced for $-1 \leq t \leq 1$. [6 marks]

(a) Show that the two endpoints of the parameter interval give the same point. [2 marks]

$\mathbf{r}(-1) = (1,\ -1+1) = (1, 0)$. [1 mark]

$\mathbf{r}(1) = (1, 1-1) = (1, 0)$. [1 mark]

Answer: $\mathbf{r}(-1) = \mathbf{r}(1) = (1, 0)$.

(b) Find the two tangent lines at this repeated point and explain why they are different. [4 marks]

$\mathbf{r}'(t) = (2t, 3t^2 - 1)$. [1 mark]

The derivative vectors at -1 and 1 are $(-2, 2)$ and $(2, 2)$, giving slopes -1 and 1. [1 mark]

Through $(1, 0)$ the tangents are $y = 1 - x$ and $y = x - 1$. [1 mark]

Different parameter values give different nonparallel directions at the same point. [1 mark]

Answer: $y = 1 - x$ at $t = -1$; $y = x - 1$ at $t = 1$.

4. A particle has acceleration $\mathbf{a}(t) = (6t, -2)$ m/s^2, velocity $\mathbf{v}(0) = (1, 4)$ m/s and position $\mathbf{r}(0) = (2, -1)$ m. [7 marks]

(a) Find the velocity and position functions. [4 marks]

Integrating acceleration and using the initial velocity gives $\mathbf{v} = (3t^2 + 1, 4 - 2t)$. [2 marks]

Integrating again and using the initial position gives $\mathbf{r} = (t^3 + t + 2, 4t - t^2 - 1)$. [2 marks]

Answer: $\mathbf{v} = (3t^2 + 1, 4 - 2t)$; $\mathbf{r} = (t^3 + t + 2, 4t - t^2 - 1)$.

(b) For $t \geq 0$, find the position and velocity at the greatest height. [3 marks]

$\frac{dy}{dt} = 4 - 2t = 0$ at $t = 2$; $\frac{d^2y}{dt^2} = -2$ confirms the maximum. [1 mark]

$\mathbf{r}(2) = (12, 3)$ m. [1 mark]

$\mathbf{v}(2) = (13, 0)$ m/s. [1 mark]

Answer: At $t = 2$ s: position $(12, 3)$ m, velocity $(13, 0)$ m/s.

5. A curve is given by $x = t^2$, $y = t$. [5 marks]

(a) At $t = 0$, find the derivative vector and use it to determine the tangent line. [3 marks]

$\mathbf{r}'(t) = (2t, 1)$, so $\mathbf{r}'(0) = (0, 1)$. [1 mark]

The point is $(0, 0)$ and the nonzero derivative is vertical. [1 mark]

Hence the tangent is $x = 0$. [1 mark]

Answer: Derivative $(0, 1)$; tangent $x = 0$.

(b) Explain why an undefined value of $\frac{dy}{dx}$ at this point does not mean that the particle is at rest. [2 marks]

The ratio $\frac{dy}{dx} = \frac{1}{2t}$ is undefined at zero because $\frac{dx}{dt} = 0$. [1 mark]

However $\frac{dy}{dt} = 1$, so the velocity is nonzero and the speed is 1. [1 mark]

Answer: $\frac{dy}{dx} = \frac{1}{2t}$ has a zero denominator at 0, but speed is 1 there.

6. A particle moves on a unit circle according to $\mathbf{r}(t) = (\cos(t^2), \sin(t^2))$ for $1 \le t \le 2$ seconds; positions are in metres. [6 marks]

(a) Find the velocity and speed functions. [3 marks]

Apply the chain rule to obtain $\mathbf{v} = (-2t\sin(t^2), 2t\cos(t^2))$. [2 marks]

Its magnitude is $2|t| = 2t$ on this interval. [1 mark]

Answer: $\mathbf{v} = (-2t\sin(t^2), 2t\cos(t^2))$ m/s; speed $2t$ m/s.

(b) Find the distance travelled from $t = 1$ to $t = 2$, and the rate of change of speed. [3 marks]

$D = \int_1^2 2t\,dt = [t^2]_1^2 = 3$ m. [2 marks]

The derivative of speed $2t$ is 2 m/s^2. [1 mark]

Answer: Distance 3 m; speed increases at 2 m/s^2.

7. A cubic Bézier curve is
$\mathbf{B}(u) = (1-u)^3P_0 + 3u(1-u)^2P_1 + 3u^2(1-u)P_2 + u^3P_3$
where $0 \le u \le 1$ and $P_0 \neq P_1$, $P_2 \neq P_3$. [6 marks]

(a) Prove that $\mathbf{B}'(0) = 3(P_1 - P_0)$. [3 marks]

The derivatives of the four scalar weights are $-3(1-u)^2$, $3(1-u)^2 - 6u(1-u)$, $6u(1-u) - 3u^2$, and $3u^2$. [2 marks]

At zero these are $-3, 3, 0, 0$, so $\mathbf{B}'(0) = 3(P_1 - P_0)$. [1 mark]

Answer: $\mathbf{B}'(0) = 3(P_1 - P_0)$.

(b) By evaluating the same derivative at $u = 1$, establish the final tangent direction. [3 marks]

At one, the four derivative weights are $0, 0, -3, 3$. [1 mark]

Therefore $\mathbf{B}'(1) = 3(P_3 - P_2)$. [1 mark]

Since $P_2 \neq P_3$, this nonzero vector points from P_2 to P_3. [1 mark]

Answer: $\mathbf{B}'(1) = 3(P_3 - P_2)$; the final direction is from P_2 towards P_3.

8. A cleaning robot follows $\mathbf{r}(t) = (3\sin t, 3\cos t)$ metres, $0 \le t \le 3\pi$, with time in seconds. [6 marks]

(a) Find the total distance travelled. [3 marks]

$\mathbf{v} = (3\cos t,\ -3\sin t)$, with speed 3 m/s. [1 mark]

$D = \int_0^{3\pi} 3\,dt = 9\pi$ m. [2 marks]

Answer: 9π m.

(b) Find the displacement and explain why its magnitude differs from the distance. [3 marks]

The endpoints are $(0, 3)$ and $(0,\ -3)$. [1 mark]

The displacement is $(0,\ -6)$, of magnitude 6 m. [1 mark]

The curved journey includes one and a half revolutions; displacement uses only the endpoints. [1 mark]

Answer: Displacement $(0,\ -6)$ m; magnitude 6 m. The robot travels one and a half revolutions.

9. Let $\mathbf{r}(t)$ be twice differentiable on an interval, with velocity $\mathbf{v}$ and acceleration $\mathbf{a}$. [6 marks]

(a) Prove that constant speed implies $\mathbf{v} \cdot \mathbf{a} = 0$. [3 marks]

$|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$ is constant. [1 mark]

Differentiating componentwise gives $0 = 2\mathbf{v} \cdot \mathbf{a}$. [1 mark]

Hence $\mathbf{v} \cdot \mathbf{a} = 0$. [1 mark]

Answer: Constant $|\mathbf{v}|$ implies $\mathbf{v} \cdot \mathbf{a} = 0$.

(b) If $\mathbf{v} \cdot \mathbf{a} = 0$ at only one instant, must speed be constant throughout the interval? Justify with a counterexample. [3 marks]

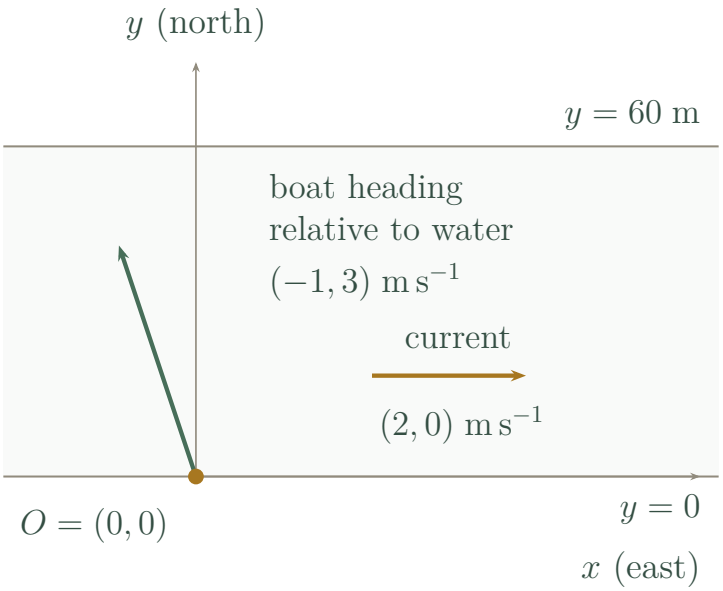
Take $\mathbf{r}(t) = (t, t^2)$, so $\mathbf{v} = (1, 2t)$ and $\mathbf{a} = (0, 2)$. [1 mark]

$\mathbf{v} \cdot \mathbf{a} = 4t = 0$ at $t = 0$. [1 mark]

Yet speed is $\sqrt{1 + 4t^2}$, which varies with t. A condition at one instant does not establish constancy on an interval. [1 mark]

Answer: No. For $\mathbf{r}(t) = (t, t^2)$, speed has zero derivative at $t = 0$ but is not constant.

10. A straight river has banks $y=0$ and $y=60$ metres, with x measured east and y north. A boat starts at the origin. The current is $(2,0)$ m/s. Initially the boat maintains velocity $(-1,3)$ m/s relative to the water. [9 marks]



(a) Find its velocity relative to the ground and where it first reaches the opposite bank. [4 marks]

Velocities add: $\mathbf{v}_g = (-1,3)+(2,0)=(1,3)$. [1 mark]

Thus $\mathbf{r}(t)=(t,3t)$. [1 mark]

$3t=60$ gives $t=20$ s and $x=20$ m. [2 marks]

Answer: Ground velocity $(1,3)$ m/s; arrival at $(20,60)$ m after 20 s.

(b) The boat can instead choose a different constant heading, while keeping its speed relative to the water equal to $\sqrt{10}$ m/s. Find the heading vector and crossing time needed to arrive directly opposite its start. [5 marks]

Zero eastward drift requires the relative-water x-component to be -2 m/s. [1 mark]

If its northward component is w, then $(-2)^2+w^2=10$, so $w=\sqrt{6}>0$. [2 marks]

The ground velocity is $(0,\sqrt{6})$ m/s. [1 mark]

The crossing time is $\frac{60}{\sqrt{6}}=10\sqrt{6}$ s. [1 mark]

Answer: Relative-water velocity $(-2,\sqrt{6})$ m/s; crossing time $10\sqrt{6}$ s.