Vector calculus - Test 1

10 questions | 58 marks | 90 minutes

Foundations, interpretation and increasingly connected motion problems. Scientific calculator permitted; no graphing, CAS or numerical-integration software is required. Show working and give exact answers unless a decimal approximation is requested. Angles are in radians. Vector coordinates use the same Cartesian axes throughout each question.

Equivalent correct methods earn the corresponding marks.

1. A curve is given by $x = t - 1$, $y = 2t^2$, for $0 \le t \le 2$. [5 marks]

(a) Eliminate the parameter and state the restriction on the Cartesian equation. [3 marks]

$t = x + 1$, hence $y = 2(x + 1)^2$. [2 marks]

The parameter interval gives $-1 \le x \le 1$. [1 mark]

Answer: $y = 2(x + 1)^2$, $-1 \le x \le 1$.

(b) State the starting point, finishing point and direction of tracing. [2 marks]

At $t = 0, 2$, the points are $(-1, 0)$ and $(1, 8)$. [1 mark]

Since $\frac{dx}{dt} = 1 > 0$, the tracing is once from left to right. [1 mark]

Answer: From $(-1, 0)$ to $(1, 8)$, once from left to right.

2. A delivery robot has position $\mathbf{r}(t) = (-2 + 3t, 3 - t)$ metres, where $t \ge 0$ is time in seconds. A painted line has equation $y = 1$. [6 marks]

(a) Find when and where the robot crosses the painted line. [3 marks]

$3 - t = 1$ gives $t = 2$ s. [2 marks]

$x = -2 + 3(2) = 4$, so the crossing point is $(4, 1)$ m. [1 mark]

Answer: $t = 2$ s, at $(4, 1)$ m.

(b) Find its speed and the distance travelled before the crossing. [3 marks]

$\mathbf{v} = (3, -1)$, so speed is $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ m/s. [2 marks]

The distance in 2 s is $2\sqrt{10}$ m. [1 mark]

Answer: Speed $\sqrt{10}$ m/s; distance $2\sqrt{10}$ m.

3. A particle has position $\mathbf{r}(t) = (t^2, 3t - t^2)$ metres for $0 \leq t \leq 3$, with t in seconds. [6 marks]

(a) Find the velocity and acceleration functions. [2 marks]

Differentiate each position component: $\mathbf{v} = (2t, 3 - 2t)$. [1 mark]

Differentiate again: $\mathbf{a} = (2, -2)$. [1 mark]

Answer: $\mathbf{v} = (2t, 3 - 2t)$ m/s; $\mathbf{a} = (2, -2)$ m/s^2.

(b) Find the greatest height above the x-axis and explain why the particle is not at rest there. [4 marks]

$\frac{dy}{dt} = 3 - 2t = 0$ gives $t = \frac{3}{2}$; $\frac{d^2y}{dt^2} = -2 < 0$, and the endpoint heights are zero. [2 marks]

$y(\frac{3}{2}) = \frac{9}{4}$ m. [1 mark]

$\mathbf{v}(\frac{3}{2}) = (3, 0) \neq \mathbf{0}$, so zero vertical velocity does not mean rest. [1 mark]

Answer: Height $\frac{9}{4}$ m at $t = \frac{3}{2}$ s; velocity $(3, 0)$ m/s is nonzero.

4. A particle has velocity $\mathbf{v}(t) = (4t, 3)$ m/s and initial position $\mathbf{r}(0) = (-1, 2)$ m. [6 marks]

(a) Find its position function. [3 marks]

Integrating gives $\mathbf{r} = (2t^2 + C_1, 3t + C_2)$. [2 marks]

The initial position gives $C_1 = -1$, $C_2 = 2$. [1 mark]

Answer: $\mathbf{r}(t) = (2t^2 - 1, 3t + 2)$ m.

(b) Find the displacement and its magnitude from $t = 0$ to $t = 2$ seconds. [3 marks]

$\mathbf{r}(2) = (7, 8)$ and $\mathbf{r}(0) = (-1, 2)$. [1 mark]

The displacement is $(8, 6)$ m. [1 mark]

Its magnitude is $\sqrt{8^2 + 6^2} = 10$ m. [1 mark]

Answer: Displacement $(8, 6)$ m; magnitude 10 m.

5. A quadratic Bézier curve has control points $P_0 = (0, 0)$, $P_1 = (1, 2)$, $P_2 = (4, 0)$. Its rule is $\mathbf{B}(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$ where $0 \le u \le 1$. [6 marks]

(a) Find $\mathbf{B}(\frac{1}{2})$ and determine whether it equals P_1. [3 marks]

$\mathbf{B}(\frac{1}{2}) = \frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{1}{4}P_2$. [1 mark]

This gives $(\frac{3}{2}, 1) \neq (1, 2)$. [2 marks]

Answer: $\mathbf{B}(\frac{1}{2}) = (\frac{3}{2}, 1)$, which is not P_1.

(b) Find the derivative vectors at both endpoints. [3 marks]

$\mathbf{B}(u) = (2u + 2u^2, 4u - 4u^2)$, so $\mathbf{B}'(u) = (2 + 4u, 4 - 8u)$. [1 mark]

At $u = 0$ the derivative is $(2, 4)$. [1 mark]

At $u = 1$ the derivative is $(6, -4)$. [1 mark]

Answer: $\mathbf{B}'(0) = (2, 4)$; $\mathbf{B}'(1) = (6, -4)$.

6. A curve has coordinates $x = t^2 + 1$, $y = t^3 - t$. [5 marks]

(a) Find the point and tangent gradient at $t = 1$. [3 marks]

At $t = 1$, $(x, y) = (2, 0)$. [1 mark]

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2 - 1}{2t}$ when $t \neq 0$, giving gradient 1 at $t = 1$. [2 marks]

Answer: Point $(2, 0)$; gradient 1.

(b) Find a Cartesian equation of that tangent. [2 marks]

The point-gradient equation is $y - 0 = 1(x - 2)$. [1 mark]

Hence $y = x - 2$. [1 mark]

Answer: $y = x - 2$.

7. A particle follows $\mathbf{r}(t) = (t^2, 2t)$ metres for $t \geq 0$ seconds. [5 marks]

(a) Find its speed function. [2 marks]

$\mathbf{v} = (2t, 2)$. [1 mark]

$s = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$. [1 mark]

Answer: $s(t) = 2\sqrt{t^2 + 1}$ m/s.

(b) Find the rate of change of speed at $t = 1$. State whether it is speeding up then. [3 marks]

$s'(t) = \frac{2t}{\sqrt{t^2 + 1}}$. [1 mark]

$s'(1) = \sqrt{2}$ m/s^2. [1 mark]

The positive value means that speed is increasing. [1 mark]

Answer: $s'(1) = \sqrt{2}$ m/s^2; it is speeding up.

8. A particle has position $\mathbf{r}(t) = (1 + 3\cos(2t),\ -2 + 3\sin(2t))$ metres, with t in seconds. [6 marks]

(a) Describe the circular path, its direction of motion and the time for one revolution. [4 marks]

$(x - 1)^2 + (y + 2)^2 = 9$, so the centre is $(1,\ -2)$ and radius is 3 m. [2 marks]

The angle $2t$ increases, so motion is anticlockwise. [1 mark]

$2T = 2\pi$ gives $T = \pi$ s. [1 mark]

Answer: Centre $(1,\ -2)$ m, radius 3 m; anticlockwise; period π s.

(b) Find its speed. [2 marks]

$\mathbf{v} = (-6\sin(2t), 6\cos(2t))$. [1 mark]

$|\mathbf{v}| = 6\sqrt{\sin^2(2t) + \cos^2(2t)} = 6$ m/s. [1 mark]

Answer: 6 m/s.

9. A curve is parameterised by $\mathbf{r}(t) = (t^2, \frac{2}{3}t^3)$, $0 \le t \le 1$. [6 marks]

(a) Show that its arc-length integrand simplifies to a function without an absolute-value sign on this interval. [3 marks]

$\mathbf{r}' = (2t, 2t^2)$. [1 mark]

$|\mathbf{r}'| = 2|t|\sqrt{1+t^2}$. [1 mark]

Since $t \ge 0$, this is $2t\sqrt{1+t^2}$. [1 mark]

Answer: $|\mathbf{r}'(t)| = 2t\sqrt{1+t^2}$ for $0 \le t \le 1$.

(b) Find the exact length of the curve. [3 marks]

$L = \int_0^1 2t\sqrt{1+t^2}\,dt$. [1 mark]

An antiderivative is $\frac{2}{3}(1+t^2)^{\frac{3}{2}}$. [1 mark]

Therefore $L = \frac{2}{3}(2\sqrt{2} - 1)$. [1 mark]

Answer: $\frac{2}{3}(2\sqrt{2} - 1)$ units.

10. A camera moves above a horizontal stage with position $\mathbf{r}(t) = (2t, 6t - t^2)$ metres, $0 \le t \le 6$, where t is measured in seconds. [7 marks]

(a) Find when the camera reaches its greatest height, and find its position then. [3 marks]

$\frac{\mathrm{d}y}{\mathrm{d}t} = 6 - 2t = 0$ gives $t = 3$; the quadratic opens downwards and both endpoint heights are zero. [2 marks]

$\mathbf{r}(3) = (6, 9)$ m. [1 mark]

Answer: $t = 3$ s; position $(6, 9)$ m.

(b) Prove that the camera has its minimum speed at that same time, and find this speed. [4 marks]

$\mathbf{v} = (2, 6 - 2t)$, so $s^2 = 4 + (6 - 2t)^2$. [1 mark]

The square is nonnegative, and is zero exactly when $t = 3$. [1 mark]

Thus $s^2 \ge 4$, so $s \ge 2$, with equality at $t = 3$. [1 mark]

The minimum speed is 2 m/s; the camera continues horizontally. [1 mark]

Answer: Minimum speed 2 m/s at $t = 3$ s.