Rates of change and differential equations - Test 6

10 questions | 63 marks | 75 minutes

Combine methods and critique common modelling errors.

Equivalent correct methods earn the corresponding marks.

1. The curve $x^2y + xy^2 = 6$ passes through $(1, 2)$. [6 marks]

(a) Find $\frac{dy}{dx}$. [3 marks]

Differentiate to $2xy + x^2y' + y^2 + 2xyy' = 0$.

Answer: $y' = -\frac{2xy + y^2}{x^2 + 2xy}$

(b) Find the tangent at the stated point. [3 marks]

At $(1, 2)$, the numerator is 8 and denominator 5.

Use point-gradient form.

Answer: $y - 2 = -\frac{8}{5}(x - 1)$

2. A rectangle has side x and perimeter 30 cm. Its x side increases at 1 cm/s. At $x = 4$ cm: [6 marks]

(a) Write a relation suitable for differentiation. [3 marks]

The other side is $15 - x$.

Answer: $A = x(15 - x)$

(b) Find the area rate in square centimetres per second. [3 marks]

$\frac{dA}{dt} = (15 - 2x)\frac{dx}{dt} = (15 - 8)(1)$.

The positive sign means that the area is increasing.

Answer: $\frac{dA}{dt} = 7$ cm^2/s

3. Two proposed solutions are $f(x) = 2e^x$ and $g(x) = e^{2x}$. The initial-value problem is $y' = 2y$, $y(0) = 2$. [6 marks]

(a) Explain which requirement each proposed function fails. [3 marks]

$f(0) = 2$, but $f' = 2e^x$ whereas $2f = 4e^x$.

$g' = 2e^{2x} = 2g$, but $g(0) = 1$.

Answer: f meets the initial condition but fails the differential equation. g meets the differential equation but fails the initial condition.

(b) Find the correct solution and verify both requirements. [3 marks]

The equation gives $y = Ae^{2x}$ and the initial value sets $A = 2$.

Then $y' = 4e^{2x} = 2y$ and $y(0) = 2$.

Answer: $y = 2e^{2x}$.

4. The equation is $y' = 1 - y^2$. A numerical sketch begins at $y(0) = 0$. [6 marks]

(a) Identify and classify the constant equilibria using the slope signs. [3 marks]

The slope vanishes at $y = \pm 1$.

It is negative below -1, positive between -1 and 1, and negative above 1.

Answer: $y = -1$ is unstable and $y = 1$ is stable.

(b) A sketch crosses $y = 1$ with positive slope. Explain the error, and verify the proposed exact solution $y = \frac{e^{2x} - 1}{e^{2x} + 1}$. [3 marks]

At $y = 1$ the equation requires slope zero, so a positive-slope crossing is inconsistent.

Quotient differentiation gives $y' = \frac{4e^{2x}}{(e^{2x} + 1)^2}$.

Also $1 - y^2 = \frac{4e^{2x}}{(e^{2x} + 1)^2}$, and $y(0) = 0$.

Answer: The proposed solution satisfies the equation and initial condition, and lies strictly between -1 and 1 for finite x.

5. A particle has acceleration $a(t) = 6t - 6$ m/s^2, $v(0) = 0$ and $x(0) = 5$ m for $0 \le t \le 3$ s. [6 marks]

(a) Find its velocity and position. [3 marks]

Integrating acceleration gives $v = 3t^2 - 6t + C_1$; $v(0) = 0$ sets $C_1 = 0$.

Integrate again and apply $x(0) = 5$.

Answer: $v = 3t^2 - 6t$ m/s; $x = t^3 - 3t^2 + 5$ m.

(b) Find its displacement and total distance during the three seconds. [3 marks]

The velocity changes sign at $t = 2$ in the interior.

$x(0) = 5$, $x(2) = 1$ and $x(3) = 5$.

Displacement is $5 - 5 = 0$; distance is $|1 - 5| + |5 - 1| = 8$.

Answer: Displacement 0 m; total distance 8 m.

6. Consider $y' = \frac{y}{x+1}$, $y(0) = -2$, $x > -1$. [6 marks]

(a) Find the solution and its interval containing the initial point. [3 marks]

Separate $\frac{dy}{y} = \frac{dx}{x+1}$, giving $\ln|y| = \ln(x+1) + C$.

Retain a signed multiplier: $y = A(x+1)$. The condition gives $A = -2$; the original equation requires $x \neq -1$.

Answer: $y = -2(x+1)$, on $x > -1$.

(b) Identify any constant solution lost during separation, or explain why there is none. [3 marks]

A zero function solves the equation on its permitted x domain.

Answer: $y = 0$

7. A reservoir contains V litres and satisfies $V' = 4 - 0.1V$, $V(0) = 10$, with t in minutes. The inflow is 4 L/min and the outflow rate is $0.1V$ L/min. [6 marks]

(a) Find the amount model. [3 marks]

Shift to $V - 40$, which decays exponentially.

Answer: $V = 40 - 30e^{-0.1t}$

(b) Find the first time the volume reaches 30 L. If the reservoir capacity is 35 L, explain whether this unchanged model can remain valid for all nonnegative time. [3 marks]

$30 = 40 - 30e^{-0.1t}$ gives $e^{-0.1t} = \frac{1}{3}$ and $t = 10\ln 3$ min. [1 mark]

The amount increases towards 40 L. Setting $35 = 40 - 30e^{-0.1t}$ gives the capacity time $t = 10\ln 6$ min. [1 mark]

After that time the formula predicts more than 35 L; overflow or changed operating conditions require a new model, or the time domain must end at capacity. [1 mark]

Answer: $t = 10\ln 3 \approx 10.99$ min. The model reaches 35 L at $10\ln 6 \approx 17.92$ min and exceeds capacity afterwards, so it cannot remain valid unchanged for all nonnegative time.

8. A population satisfies $P' = 0.2P(1 - \frac{P}{900})$ with $P(0) = 600$. Here t is in days, P is in individuals, and t is at least zero. [9 marks]

(a) Find $P(t)$ and the carrying capacity. [3 marks]

$A = \frac{900 - 600}{600} = 0.5$.

Answer: $P = \frac{900}{1 + 0.5e^{-0.2t}}$; carrying capacity 900.

(b) Find the global largest possible positive growth rate and when this trajectory reaches its associated amount. [3 marks]

The quadratic rate has its vertex at $P = \frac{K}{2}$, with value $\frac{rK}{4}$.

The maximum positive growth along this future trajectory is initially; $\frac{K}{2}$ occurred at a negative model time.

Answer: The global rate maximum is 45 individuals/day at $P = 450$. This trajectory never reaches $P = 450$ for nonnegative t.

(c) For this initial population, find the greatest growth rate for t at least zero. [3 marks]

The trajectory starts at $P = 600$, already above $\frac{K}{2} = 450$.

For P above $\frac{K}{2}$ the logistic rate decreases as P increases.

The initial rate is $0.2(600)(1 - \frac{600}{900}) = 40$.

Answer: 40 individuals/day at $t = 0$.

9. A displacement model is $x = 2\cos 3t - 2\sin 3t$. Here t is measured in seconds and displacement x in metres. [6 marks]

(a) Differentiate the given displacement to find velocity and acceleration, and verify that $a = -9x$. [3 marks]

Differentiate once: $v(t) = -6\sin(3t) - 6\cos(3t)$. [1 mark]

Differentiate again: $a(t) = -18\cos(3t) + 18\sin(3t)$. [1 mark]

Factor the acceleration: $a(t) = -9(2\cos(3t) - 2\sin(3t)) = -9x(t)$. [1 mark]

Answer: $v = -6\sin(3t) - 6\cos(3t)$ m/s; $a = -18\cos(3t) + 18\sin(3t) = -9x$ m/s^2.

(b) Find the amplitude and maximum speed. [3 marks]

Use amplitude $\sqrt{4+4}$, then multiply by 3.

Answer: Amplitude $2\sqrt{2}$; maximum speed $6\sqrt{2}$.

10. For $y' = x - y$, a student calls $y = x$ an equilibrium because its field slopes vanish. [6 marks]

(a) Identify and correct the error. [3 marks]

Along $y = x$, the derivative of the proposed function is 1, but the differential equation requires $x - x = 0$.

Zero slopes along a nonhorizontal line do not make that line a solution.

Answer: This is the zero-slope line, not an equilibrium solution.

(b) Prove that no constant equilibrium exists. [3 marks]

If $y = c$ for a constant c, then $y' = 0$.

The right side is $x - c$, which cannot be zero for every x in an interval.

Answer: No constant equilibrium exists.