Rates of change and differential equations - Test 5

10 questions | 60 marks | 75 minutes

Apply calculus while checking signs, distances and assumptions.

Equivalent correct methods earn the corresponding marks.

1. The curve $x^3 + y^3 = 9$ passes through $(1, 2)$. [6 marks]

(a) Find $\frac{dy}{dx}$. [3 marks]

$3x^2 + 3y^2y' = 0.$

Answer: $y' = -\frac{x^2}{y^2}$

(b) Find the tangent at the stated point. [3 marks]

At $(1, 2)$, gradient is $-\frac{1}{4}$.

Use point-gradient form.

Answer: $y - 2 = -\frac{1}{4}(x - 1)$

2. Water leaves a cylinder of fixed radius 3 m at 6 m^3/min. Let h be its depth. [6 marks]

(a) Write a relation suitable for differentiation. [3 marks]

The base area is fixed.

Answer: $V = 9\pi h$

(b) Find the depth rate in metres per minute. [3 marks]

$-6 = 9\pi \frac{dh}{dt}.$

Answer: $\frac{dh}{dt} = -\frac{2}{3\pi}$ m/min

3. A student claims the general solution of $y' = 3y$ is $y = e^{3x+C}$ and therefore the initial condition $y(0) = -2$ is impossible. [6 marks]

(a) Explain the missing step and give the full solution family. [3 marks]

For nonzero y, separation gives $\ln|y| = 3x + C$.

Exponentiating the absolute value requires a signed multiplier; the equilibrium $y = 0$ is checked before dividing.

Answer: $y = Ae^{3x}$ for any real A, including $A = 0$.

(b) Find and verify the solution with $y(0) = -2$. [3 marks]

The initial value sets $A = -2$.

$y' = -6e^{3x} = 3y$, and $y(0) = -2$.

Answer: $y = -2e^{3x}$.

4. Consider the slope field for $y' = -y^3$. [6 marks]

(a) Find slopes at $(0, -1)$ and $(2, 2)$. [3 marks]

Cube y, then negate.

Answer: 1 and -8

(b) Find and classify constant equilibria, or explain why none exist. [3 marks]

Negative y gives positive slope and positive y gives negative slope.

Answer: $y = 0$, stable

5. Solve the initial-value problem $y' = \frac{2x}{1+x^2}$, $y(0) = 3$. [6 marks]

(a) Find $y(x)$. [3 marks]

The numerator is the derivative of the denominator; $C = 3$.

Answer: $y = \ln(1 + x^2) + 3$

(b) Find $y(1)$ and verify your derivative. [3 marks]

Substitute $x = 1$.

Differentiate the found function to recover the equation.

Answer: $y(1) = \ln 2 + 3$

6. Consider $y' = 2x(y - 1)$ with $y(0) = -1$. [6 marks]

(a) Solve the initial-value problem and state its domain. [3 marks]

For $y \neq 1$, separate $\frac{dy}{y-1} = 2x\,dx$.

$\ln|y - 1| = x^2 + C$, so $y = 1 + Ae^{x^2}$. The condition gives $A = -2$.

Answer: $y = 1 - 2e^{x^2}$ for all real x.

(b) Find the equilibrium lost in separation. Explain whether this initial-value solution ever reaches it. [3 marks]

The original equation has zero right side when $y = 1$.

For the required solution, $y - 1 = -2e^{x^2} < 0$, so it never equals the equilibrium.

Answer: The equilibrium is $y = 1$. This solution stays below 1 for every real x.

7. A medicine concentration follows exponential decay. It starts at 12 mg/L and is 3 mg/L after 8 h. [6 marks]

(a) Find the concentration model. [3 marks]

Calibrate with $\frac{C(8)}{C(0)} = \frac{1}{4}$.

Answer: $C = 12e^{-\frac{(\ln 4)t}{8}}$

(b) Find the half-life. [3 marks]

One-quarter after 8 h is two half-lives.

Answer: Half-life 4 h

8. A population follows $P' = 0.4P(1 - \frac{P}{1000})$, $P(0) = 1200$, with t in years and $t \geq 0$. [6 marks]

(a) Find the model, initial rate and long-term limit. [3 marks]

$A = \frac{1000 - 1200}{1200} = -\frac{1}{6}$.

$P'(0) = 0.4(1200)(1 - 1.2) = -96$. As $t \to \infty$, the exponential tends to zero.

Answer: $P = \frac{1000}{1 - \frac{e^{-0.4t}}{6}}$; initial rate -96 individuals/year; limit 1000.

(b) A student says the population must first rise to 1500 before settling at 1000. Explain why the differential equation rules this out. [3 marks]

The factors $0.4P$ and $1 - \frac{P}{1000}$ have opposite signs for $P > 1000$.

The denominator lies between $\frac{5}{6}$ and 1 for future times, so the model stays between 1000 and 1200.

Answer: For every $P > 1000$ the derivative is negative, so the population decreases toward 1000 from above.

9. A particle moves according to $x'' = -4x$, with $x(0) = 3$ m and $v(0) = 8$ m/s. [6 marks]

(a) Find $x(t)$ and the amplitude. [3 marks]

Write $x = A\cos 2t + B\sin 2t$. The initial data give $A = 3$ and $2B = 8$.

The amplitude is $\sqrt{3^2 + 4^2} = 5$.

Answer: $x = 3\cos 2t + 4\sin 2t$ m; amplitude 5 m.

(b) Find the maximum speed and the first positive time at which the particle reaches its positive extreme position. [3 marks]

The angular frequency is 2, so maximum speed is $2(5) = 10$.

Write $x = 5\cos(2t - \phi)$ with $\cos\phi = \frac{3}{5}$ and $\sin\phi = \frac{4}{5}$.

The first maximum occurs at $2t = \phi = \arctan(\frac{4}{3})$.

Answer: Maximum speed 10 m/s; $t = \frac{1}{2}\arctan(\frac{4}{3}) \approx 0.464$ s.

10. A particle has velocity $v = t - 1$ m/s for $0 \le t \le 3$ s. A student calls $\int_0^3 v\,dt$ the total distance. [6 marks]

(a) Identify and correct the error. [3 marks]

Velocity is negative on $0 < t < 1$ and positive on $1 < t < 3$.

Compute $\int_0^3 |v|\,dt$ by splitting at $t = 1$.

Answer: The signed integral gives displacement; total distance uses the integral of speed.

(b) Find both displacement and total distance on the stated interval. [3 marks]

$\int_0^3 (t - 1)\,dt = [\frac{t^2}{2} - t]_0^3 = 1.5$ m.

The negative area has magnitude 0.5 m and the positive area is 2 m, so distance is 2.5 m.

Answer: Displacement 1.5 m; total distance 2.5 m.