Rates of change and differential equations - Test 4

10 questions | 60 marks | 75 minutes

Reason from functions and slope information.

Equivalent correct methods earn the corresponding marks.

1. The relation $\sin y = x$ passes through $(0, 0)$. [6 marks]

(a) Find $\frac{dy}{dx}$. [3 marks]

The chain rule gives $\cos y \, y' = 1$.

Answer: $y' = \frac{1}{\cos y}$

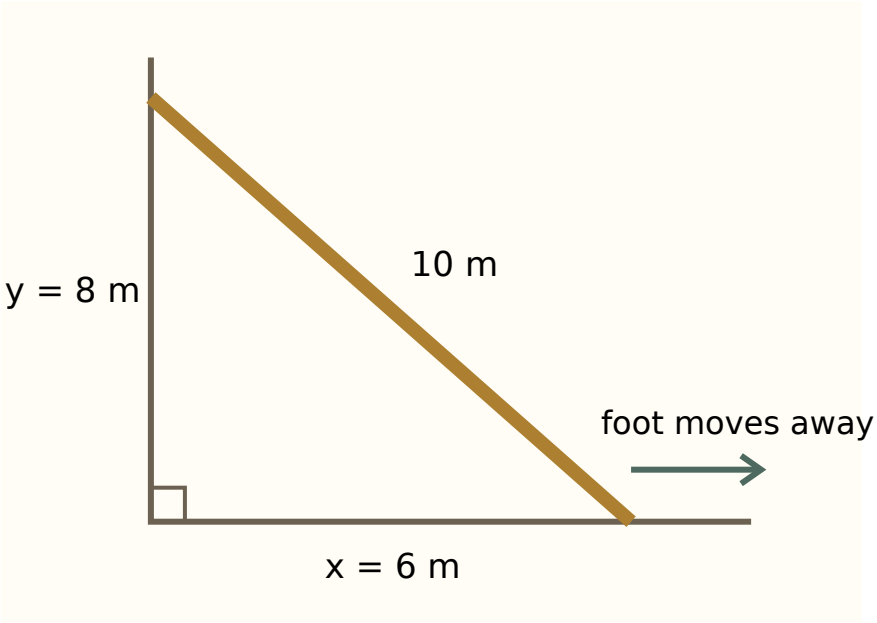
(b) Find the tangent at the stated point. [3 marks]

At $(0, 0)$, gradient is 1.

Use point-gradient form.

Answer: $y = x$

2. A 10 m ladder has its foot 6 m from a wall and moving away at 0.4 m/s. Let y be top height. [6 marks]



(a) Write a relation suitable for differentiation. [3 marks]

At the instant, $y = 8$ m.

Answer: $x^2 + y^2 = 100$

(b) Find the ladder-top height rate in metres per second. [3 marks]

$\frac{dy}{dt} = -\frac{x(\frac{dx}{dt})}{y} = -\frac{6(0.4)}{8}$.

Answer: $\frac{dy}{dt} = -0.3$ m/s

3. A proposed initial-value solution is $y = \frac{2}{1-2x}$, $x < \frac{1}{2}$. [6 marks]

(a) Verify $y' = y^2$. [3 marks]

$y' = \frac{4}{(1-2x)^2} = y^2$.

Answer: It satisfies the differential equation.

(b) Verify $y(0) = 2$. Explain why both checks are needed. [3 marks]

Substitute the initial x into the function.

The initial value selects the required family member.

Answer: It satisfies the initial condition; an equation alone normally has a family of solutions.

4. An autonomous differential equation has the form $y' = a - by$. Its slope is 6 at $y = 0$ and 2 at $y = 2$. [6 marks]

(a) Determine a and b, and find the constant equilibrium. [3 marks]

At $y = 0$, $a = 6$. At $y = 2$, $6 - 2b = 2$, so $b = 2$.

Set $6 - 2y = 0$ to obtain $y = 3$.

Answer: $a = 6$, $b = 2$; equilibrium $y = 3$.

(b) Classify the equilibrium and solve the equation with $y(0) = 1$. [3 marks]

Below 3 the slope is positive; above 3 it is negative.

Set $u = y - 3$. Then $u' = -2u$ and $u(0) = -2$, giving the stated result.

Answer: Stable; $y = 3 - 2e^{-2x}$.

5. A tank starts with 20 L. Its net inflow rate is $q(t) = 4 - t$ L/min for $0 \le t \le 4$ and $q(t) = -2$ L/min for $4 < t \le 7$. [6 marks]

(a) Find the amount at 4 minutes and at 7 minutes. [3 marks]

$V(4) = 20 + \int_0^4 (4 - t)\,dt = 20 + 8 = 28.$

$V(7) = 28 + \int_4^7 (-2)\,dt = 28 - 6 = 22.$

Answer: $V(4) = 28$ L and $V(7) = 22$ L.

(b) Find the maximum amount during the stated interval and explain why it occurs there. [3 marks]

The rate is positive before $t = 4$, so the amount increases.

The rate is negative after $t = 4$, so the amount then decreases.

Answer: 28 L at $t = 4$ min.

6. Consider $y' = (1 + y^2)$, $y(0) = 0$. [6 marks]

(a) Find the solution and its interval containing the initial point. [3 marks]

Integrate $\frac{dy}{1+y^2} = dx$ to $\arctan y = x + C$.

Answer: $y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) Identify any constant solution lost during separation, or explain why there is none. [3 marks]

$1 + y^2$ is positive for every real y.

Answer: No real constant equilibrium

7. A laboratory compares two amount models for $t \geq 0$ days: $A(t) = 100e^{0.1t}$ and $B(t) = 100 + 10t$. [6 marks]

(a) Show that they have the same initial amount and the same initial rate. Find each amount at 10 days. [3 marks]

$A(0) = B(0) = 100$. Also $A' = 10e^{0.1t}$ and $B' = 10$, so both initial rates are 10.

Substitute $t = 10$ into both models.

Answer: Both start at 100 with initial rate 10 units/day. $A(10) = 100e \approx 271.83$ and $B(10) = 200$.

(b) Explain why equal initial values and rates do not make these models identical. [3 marks]

The equations are $A' = 0.1A$ and $B' = 10$.

For $t > 0$, the exponential model has an increasing rate while the linear model keeps its initial rate.

Answer: A has a rate proportional to its current amount; B has a constant rate.

8. A logistic population satisfies $P' = rP(1 - \frac{P}{1000})$. When $P = 200$, its rate is 32 individuals/day. Initially $P(0) = 100$. [6 marks]

(a) Find r and the population model. [3 marks]

$32 = r(200)(1 - \frac{200}{1000}) = 160r$, giving $r = 0.2$.

$A = \frac{1000 - 100}{100} = 9$ in the logistic solution.

Answer: $r = 0.2$ day^{-1}; $P = \frac{1000}{1 + 9e^{-0.2t}}$.

(b) Find the largest growth rate and the time it occurs. [3 marks]

The rate is greatest at $P = 500$, with value $\frac{rK}{4} = 50$.

Solve $500 = \frac{1000}{1 + 9e^{-0.2t}}$: $9e^{-0.2t} = 1$, hence $t = 5\ln 9$.

Answer: 50 individuals/day at $t = 5\ln 9 \approx 10.99$ days.

9. A displacement model is $x'' = -16(x-2)$, $x(0)=3$, $v(0)=0$. Here t is measured in seconds and displacement x in metres. [6 marks]

(a) Find or verify the displacement and its derivative relation. [3 marks]

Shift $u = x - 2$ and use initial $u = 1$.

Answer: $x = 2 + \cos 4t$

(b) Find the equilibrium position and period. [3 marks]

Angular frequency 4.

Answer: Equilibrium 2; period $\frac{\pi}{2}$.

10. A 10 m ladder slides with its foot at distance x from a wall and its top at height y. A student substitutes $x=6$, $y=8$ into $x^2+y^2=100$ before differentiating. [6 marks]

(a) Identify and correct the error. [3 marks]

Keep x and y as functions of time and obtain $2x\dot{x} + 2y\dot{y} = 0$.

Only then insert the instantaneous dimensions; differentiating a numerical identity gives no useful rate relation.

Answer: Keeping dimensions variable is necessary; otherwise differentiating constants gives no rate relation.

(b) At that instant, find the top rate if the foot moves away at 0.2 m/s. [3 marks]

$\dot{y} = -\frac{x\dot{x}}{y} = -\frac{6(0.2)}{8} = -0.15$ m/s.

The top is moving down.

Answer: $\dot{y} = -0.15$ m/s.