Rates of change and differential equations - Test 3

10 questions | 60 marks | 75 minutes

Connect geometry, separation and rate models.

Equivalent correct methods earn the corresponding marks.

1. The relation $xy + y^2 = 6$ passes through $(1, 2)$. [6 marks]

(a) Find $\frac{dy}{dx}$. [3 marks]

Use the product rule: $y + xy' + 2yy' = 0$.

Answer: $y' = -\frac{y}{x + 2y}$

(b) Find the tangent at the stated point. [3 marks]

Substitution gives gradient $-\frac{2}{5}$.

Use point-gradient form.

Answer: $y - 2 = -\frac{2}{5}(x - 1)$

2. A conical tank has its vertex at the bottom and radius-to-height ratio $r = \frac{h}{2}$. Water enters at 3 m^3/min. At depth $h = 2$ m: [6 marks]

(a) Write a relation suitable for differentiation. [3 marks]

Substitute $r = \frac{h}{2}$ into $V = \frac{\pi r^2 h}{3}$.

Answer: $V = \frac{\pi h^3}{12}$

(b) Find the water-depth rate in metres per minute. [3 marks]

$\frac{dV}{dt} = (\frac{\pi h^2}{4})\frac{dh}{dt}$; at $h = 2$, the multiplier is π.

Answer: $\frac{dh}{dt} = \frac{3}{\pi}$ m/min

3. The table shows values of a differentiable function satisfying $y' = 2x$. [6 marks]

Observed values

x	y
0	3
1	4
2	7
3	12

(a) Use the initial value in the table to find the function. [3 marks]

Integrating gives $y = x^2 + C$.

The entry $y(0) = 3$ gives $C = 3$.

Answer: $y = x^2 + 3$.

(b) Check whether all the table entries are consistent with this model. Does agreement with a finite table alone prove a differential equation for all real x? [3 marks]

The model predicts $3, 4, 7, 12$ at $x = 0, 1, 2, 3$.

The equation is verified for all x by differentiating $x^2 + 3$, rather than by checking finitely many values.

Answer: All four entries agree. A finite table alone does not establish the derivative relation everywhere.

4. Consider the slope field for $y' = y(y - 2)$. [6 marks]

(a) Find slopes at $(0, 1)$ and $(1, 3)$. [3 marks]

Evaluate $y(y - 2)$.

Answer: -1 and 3

(b) Find and classify constant equilibria, or explain why none exist. [3 marks]

Slopes are positive below 0, negative between 0 and 2, positive above 2.

Answer: $y = 0$ stable and $y = 2$ unstable

5. Solve the initial-value problem $y' = \frac{3}{x+1}$, $y(0) = 2$, $x > -1$. [6 marks]

(a) Find $y(x)$. [3 marks]

Integrate the reciprocal linear function; $C = 2$.

Answer: $y = 3\ln(x+1) + 2$

(b) Find $y(e-1)$ and verify your derivative. [3 marks]

$x + 1 = e$, so the logarithm is 1.

Differentiate the found function to recover the equation.

Answer: $y(e-1) = 5$

6. Consider $y' = \frac{x}{y}$, $y(0) = 2$. [6 marks]

(a) Find the solution and its interval containing the initial point. [3 marks]

$y\,dy = x\,dx$; $y^2 = x^2 + 4$, positive initial branch.

Answer: $y = \sqrt{x^2+4}$, for all real x.

(b) Identify any constant solution lost during separation, or explain why there is none. [3 marks]

The original equation is undefined at $y = 0$.

A nonzero constant $y = c$ would require $0 = \frac{x}{c}$ for every x, which is impossible.

Answer: There is no constant solution.

7. An object cools according to $T' = -0.2(T - 20)$, with $T(0) = 80$. Temperature is in degrees Celsius and time in minutes. [6 marks]

(a) Find the temperature model. [3 marks]

Let $u = T - 20$. Then $u' = -0.2u$ and $u(0) = 60$.

Hence $u = 60e^{-0.2t}$ and $T = 20 + 60e^{-0.2t}$.

Answer: $T = 20 + 60e^{-0.2t}$

(b) Find when the temperature first reaches 40 degrees Celsius. [3 marks]

$40 = 20 + 60e^{-0.2t}$ gives $e^{-0.2t} = \frac{1}{3}$.

Taking logarithms gives $t = \frac{\ln 3}{0.2}$.

Answer: $t = 5\ln 3 \approx 5.49$ min.

8. A population satisfies $P' = 0.3P(1 - \frac{P}{800})$ with $P(0) = 200$. Here t is in days, P is in individuals, and t is at least zero. [6 marks]

(a) Find $P(t)$ and the carrying capacity. [3 marks]

$A = \frac{800 - 200}{200} = 3$.

Answer: $P = \frac{800}{1 + 3e^{-0.3t}}$; carrying capacity 800.

(b) Find the global largest possible positive growth rate and when this trajectory reaches its associated amount. [3 marks]

The quadratic rate has its vertex at $P = \frac{K}{2}$, with value $\frac{rK}{4}$.

Set $P = \frac{K}{2}$ to make the denominator 2, so $Ae^{-rt} = 1$.

Answer: Rate 60 at $P = 400$ individuals/day; $t = \frac{\ln(3)}{0.3} \approx 3.662$ days.

9. A displacement model is $x'' = -x$, $x(0) = 3$, $v(0) = 4$. Here t is measured in seconds and displacement x in metres. [6 marks]

(a) Find or verify the displacement and its derivative relation. [3 marks]

$A = 3$, $B = 4$.

Answer: $x = 3\cos t + 4\sin t$

(b) Find the amplitude and period. [3 marks]

Use $\sqrt{A^2 + B^2}$.

Answer: Amplitude 5; period 2π.

10. A logistic rate is $P' = 0.5P(1 - \frac{P}{400})$. A student says growth is greatest at $P = 400$. [6 marks]

(a) Identify and correct the error. [3 marks]

The rate is a downward quadratic $0.5P - 0.00125P^2$, whose vertex is at half the carrying capacity.

Answer: The rate is zero at $P = 400$ and greatest at $P = 200$.

(b) Calculate the greatest possible growth rate. [3 marks]

At $P = 200$, the rate is $0.5(200)(1 - \frac{200}{400}) = 50$.

Answer: 50 individuals per unit time.