Rates of change and differential equations - Test 2

10 questions | 60 marks | 75 minutes

Use different equation forms and interpret their solutions.

Equivalent correct methods earn the corresponding marks.

1. The ellipse $x^2 + 4y^2 = 20$ passes through $(2, 2)$. [6 marks]

(a) Find $\frac{dy}{dx}$. [3 marks]

$2x + 8yy' = 0$.

Answer: $y' = -\frac{x}{4y}$

(b) Find the tangent at the stated point. [3 marks]

The gradient at $(2, 2)$ is $-\frac{1}{4}$.

Use point-gradient form.

Answer: $y - 2 = -\frac{1}{4}(x - 2)$

2. A sphere radius r decreases at 0.1 cm/s. At $r = 3$ cm: [6 marks]

(a) Write a relation suitable for differentiation. [3 marks]

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Answer: $V = \frac{4}{3}\pi r^3$

(b) Find the volume rate in cubic centimetres per second and interpret its sign. [3 marks]

Use $\frac{dr}{dt} = -0.1$ cm/s: $\frac{dV}{dt} = 4\pi(3)^2(-0.1) = -3.6\pi$ cm^3/s.

The negative sign agrees with a shrinking sphere.

Answer: -3.6π cm^3/s; the volume is decreasing.

3. A proposed initial-value solution is $y = x^3 + 4$. [6 marks]

(a) Verify $y' = 3x^2$. [3 marks]

$y' = 3x^2$.

Answer: It satisfies the differential equation.

(b) Verify $y(1) = 5$. Explain why both checks are needed. [3 marks]

Substitute the initial x into the function.

The initial value selects the required family member.

Answer: It satisfies the initial condition; an equation alone normally has a family of solutions.

4. Consider the slope field for $y' = 3 - y$. [6 marks]

(a) Find slopes at $(0, 1)$ and $(2, 4)$. [3 marks]

Evaluate $3 - y$.

Answer: 2 and -1

(b) Find and classify constant equilibria, or explain why none exist. [3 marks]

Below 3 slopes are positive; above 3 negative.

Answer: $y = 3$, stable

5. Solve the initial-value problem $y' = 4e^{2x}$, $y(0) = 5$. [6 marks]

(a) Find $y(x)$. [3 marks]

Integrate with factor $\frac{1}{2}$; initial value gives 3.

Answer: $y = 2e^{2x} + 3$

(b) Find $y(\ln 2)$ and verify your derivative. [3 marks]

$e^{2\ln 2} = 4$, so $2(4) + 3 = 11$.

Differentiate the found function to recover the equation.

Answer: $y(\ln 2) = 11$

6. Consider $y' = y^2$, $y(0) = 1$. [6 marks]

(a) Find the solution and its interval containing the initial point. [3 marks]

$y^{-2}dy = dx$ gives $-\frac{1}{y} = x + C$; $C = -1$.

Answer: $y = \frac{1}{1-x}$, $x < 1$

(b) Identify any constant solution lost during separation, or explain why there is none. [3 marks]

The original equation is satisfied by $y = 0$, which is excluded when dividing by y^2.

Answer: $y = 0$

7. A sample has half-life 6 h and initial mass 90 g. [6 marks]

(a) Find the amount model. [3 marks]

Half-life requires $e^{6k} = \frac{1}{2}$.

Answer: $M = 90e^{-\frac{(\ln 2)t}{6}}$

(b) Find the mass after 12 hours. [3 marks]

Two half-lives leave one-quarter.

Answer: $M(12) = 22.5$ g

8. A population satisfies $P' = 0.2P(1 - \frac{P}{600})$ with $P(0) = 100$. Here t is in days, P is in individuals, and t is at least zero. [6 marks]

(a) Find $P(t)$ and the carrying capacity. [3 marks]

$A = \frac{600 - 100}{100} = 5$.

Answer: $P = \frac{600}{1 + 5e^{-0.2t}}$; carrying capacity 600.

(b) Find the global largest possible positive growth rate and when this trajectory reaches its associated amount. [3 marks]

The quadratic rate has its vertex at $P = \frac{K}{2}$, with value $\frac{rK}{4}$.

Set $P = \frac{K}{2}$ to make the denominator 2, so $Ae^{-rt} = 1$.

Answer: Rate 30 at $P = 300$ individuals/day; $t = \frac{\ln(5)}{0.2} \approx 8.047$ days.

9. A displacement model is $x'' = -9x$, $x(0) = 0$, $v(0) = 6$. Here t is measured in seconds and displacement x in metres. [6 marks]

(a) Find or verify the displacement and its derivative relation. [3 marks]

$A = 0$ and $3B = 6$.

Answer: $x = 2\sin 3t$

(b) Find the period and maximum speed. [3 marks]

Angular frequency 3 and amplitude 2.

Answer: Period $\frac{2\pi}{3}$; maximum speed 6.

10. A student solves $y' = y^2$, $y(0) = 1$ and reports $y = \frac{1}{1-x}$ for every real x. [6 marks]

(a) Identify and correct the error. [3 marks]

The denominator is zero at $x = 1$.

The formula also defines a separate branch for $x > 1$, but that interval does not contain the initial point.

Answer: The maximal interval containing the initial point is $x < 1$.

(b) Describe the behaviour as x approaches 1 from below. [3 marks]

This is a finite-parameter blow-up, not an equilibrium.

Answer: As x approaches 1 from below, y tends to positive infinity.