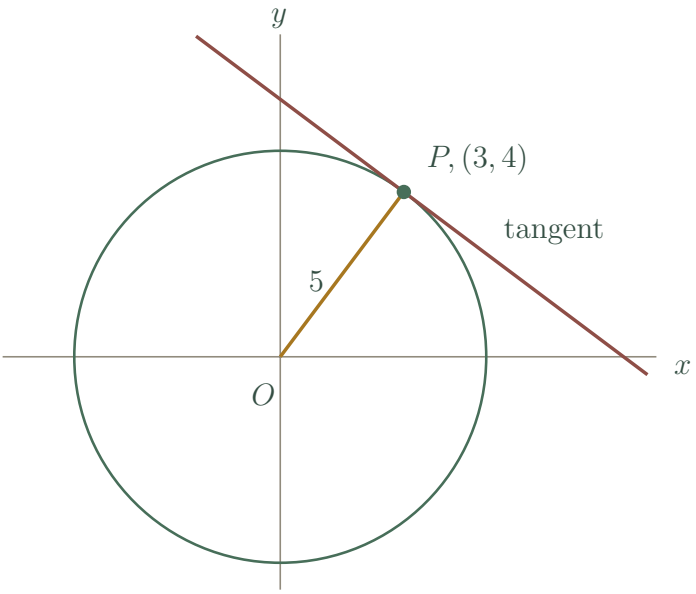
Rates of change and differential equations - Test 1

10 questions | 60 marks | 75 minutes

Build fluency across implicit rates, initial conditions and models.

Equivalent correct methods earn the corresponding marks.

1. The circle $x^2 + y^2 = 25$ passes through $(3, 4)$. [6 marks]



(a) Find $\frac{dy}{dx}$. [3 marks]

$2x + 2yy' = 0$.

Answer: $y' = -\frac{x}{y}$

(b) Find the tangent at the stated point. [3 marks]

The gradient at $(3, 4)$ is $-\frac{3}{4}$.

Use point-gradient form.

Answer: $y - 4 = -\frac{3}{4}(x - 3)$

2. A square side s increases at 2 cm/s. At $s = 5$ cm: [6 marks]

(a) Write a relation suitable for differentiation. [3 marks]

$\frac{dA}{dt} = 2s\,\frac{ds}{dt}$

Answer: $A = s^2$

(b) Find the area rate in cm squared per second. [3 marks]

$2(5)(2) = 20.$

Answer: 20 cm^2/s

3. A proposed initial-value solution is $y = 3e^{-2x}$. [6 marks]

(a) Verify $y' = -2y$. [3 marks]

$y' = -6e^{-2x} = -2y.$

Answer: It satisfies the differential equation.

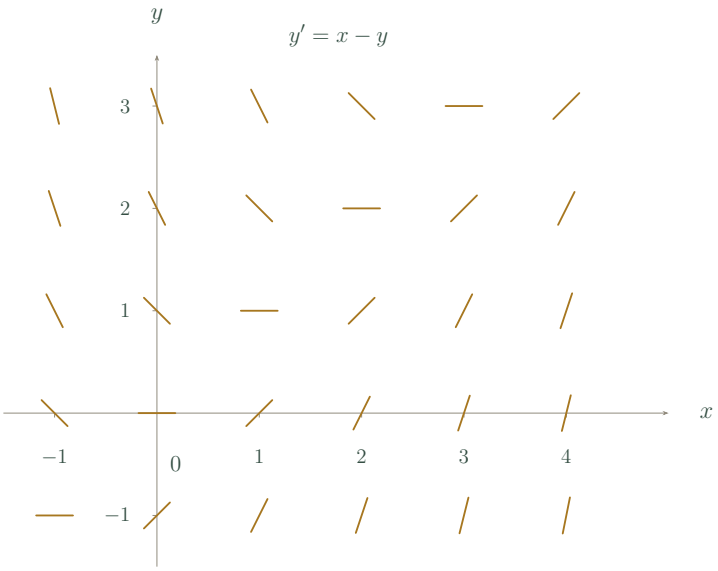
(b) Verify $y(0) = 3$. Explain why both checks are needed. [3 marks]

Substitute the initial x into the function.

The initial value selects the required family member.

Answer: It satisfies the initial condition; an equation alone normally has a family of solutions.

4. Consider the slope field for $y' = x - y$. [6 marks]



(a) Find slopes at $(-1, 1)$ and $(1, 0)$. [3 marks]

Evaluate $x - y$ at each point.

Answer: -2 and 1

(b) Identify the line of zero slopes and explain whether it is a constant equilibrium. [3 marks]

Set $x - y = 0$ to obtain $y = x$.

The line is not constant; moreover, along it the derivative of $y = x$ would be 1 rather than the required zero.

Answer: The zero-slope line is $y = x$; it is not a constant equilibrium.

5. Solve the initial-value problem $y' = 6x^2 - 2$, $y(0) = 4$. [6 marks]

(a) Find $y(x)$. [3 marks]

Integrate to $2x^3 - 2x + C$; initial value gives $C = 4$.

Answer: $y = 2x^3 - 2x + 4$

(b) Find $y(2)$ and verify your derivative. [3 marks]

$16 - 4 + 4 = 16$.

Differentiate the found function to recover the equation.

Answer: $y(2) = 16$

6. Consider $y' = 2xy$, $y(0) = 3$. [6 marks]

(a) Find the solution and its interval containing the initial point. [3 marks]

$\frac{dy}{y} = 2x\,dx$; integrate to $\ln|y| = x^2 + C$.

Answer: $y = 3e^{x^2}$, for all real x.

(b) Identify any constant solution lost during separation, or explain why there is none. [3 marks]

Check $y = 0$ before dividing by y.

Answer: $y = 0$

7. A culture grows from 200 to 300 individuals in 4 days under $P' = kP$, with constant k. [6 marks]

(a) Find the amount model. [3 marks]

$300 = 200e^{4k}$ gives $k = \frac{\ln 1.5}{4}$.

Insert k and the initial amount into $P = 200e^{kt}$.

Answer: $P(t) = 200e^{\frac{(\ln 1.5)t}{4}}$, where t is in days.

(b) Find the doubling time. [3 marks]

Set $P(T) = 400$: $400 = 200e^{kT}$, so $T = \frac{\ln 2}{k}$.

Using $k = \frac{\ln(1.5)}{4}$ gives $T = \frac{4\ln 2}{\ln(1.5)} \approx 6.84$ days.

Answer: Doubling time $\frac{4\ln 2}{\ln 1.5} \approx 6.84$ days

8. A population satisfies $P' = 0.4P(1 - \frac{P}{1000})$ with $P(0) = 200$. Here t is in days, P is in individuals, and t is at least zero. [6 marks]

(a) Find $P(t)$ and the carrying capacity. [3 marks]

$A = \frac{1000 - 200}{200} = 4$.

Answer: $P = \frac{1000}{1 + 4e^{-0.4t}}$; carrying capacity 1000.

(b) Find the global largest possible positive growth rate and when this trajectory reaches its associated amount. [3 marks]

The quadratic rate has its vertex at $P = \frac{K}{2}$, with value $\frac{rK}{4}$.

Set $P = \frac{K}{2}$ to make the denominator 2, so $Ae^{-rt} = 1$.

Answer: Rate 100 at $P = 500$ individuals/day; $t = \frac{\ln(4)}{0.4} \approx 3.466$ days.

9. A displacement model is $x'' = -4x$, $x(0) = 3$, $v(0) = 0$. Here t is measured in seconds and displacement x in metres. [6 marks]

(a) Find or verify the displacement and its derivative relation. [3 marks]

Use $x = A\cos 2t + B\sin 2t$; $A = 3$, $B = 0$.

Answer: $x = 3\cos 2t$

(b) Find the period and maximum speed. [3 marks]

Angular frequency 2 and amplitude 3.

Answer: Period π; maximum speed 6.

10. A student separates $y' = 2y$ and writes $\ln|y| = 2x + C$, then claims that y must be positive. [6 marks]

(a) Identify and correct the error. [3 marks]

From $\ln|y| = 2x + C$, $|y| = e^C e^{2x}$.

The sign of y is retained in a signed multiplier A. The equilibrium is checked before division by y.

Answer: The family is $y = Ae^{2x}$ for any real A; $A = 0$ gives the equilibrium.

(b) Find a solution meeting $y(0) = -3$ and verify it. [3 marks]

The initial value gives $A = -3$.

The derivative is $-6e^{2x} = 2y$ and the initial value is -3.

Answer: $y = -3e^{2x}$.