# Integration techniques and applications - practice test 6

10 questions | 64 marks | 75 minutes

Challenging combined-method problems, parameter reasoning, singularities and geometric models.

Equivalent correct methods earn the corresponding marks.

**1.** For $x>0$, consider $\int x^n \ln x\,dx$, where n is real. [7 marks]

**(a)** For $n \neq -1$, derive a general antiderivative using parts. [4 marks]

Take $u=\ln x$ and $dv=x^n\,dx$, so $v=\frac{x^{n+1}}{n+1}$.

The remaining integral is $\frac{1}{n+1}\int x^n\,dx$.

**Answer:** $\frac{x^{n+1}\ln x}{n+1}-\frac{x^{n+1}}{(n+1)^2}+C$.

**(b)** Handle $n=-1$ separately and evaluate $\int_1^{e^2}\frac{\ln x}{x}\,dx$. [3 marks]

Use $u=\ln x$, $du=\frac{dx}{x}$.

The transformed limits are 0 and 2; integrate $u$ from 0 to 2.

**Answer:** Antiderivative $\frac{(\ln x)^2}{2}+C$; definite value $2$.

**2.** Let $I=\int_0^1 \frac{x^3}{1+x^2}\,dx$. [6 marks]

**(a)** Use polynomial division to find an antiderivative. [3 marks]

Write $\frac{x^3}{1+x^2}=x-\frac{x}{1+x^2}$.

Integrate the first term directly and the second by $u=1+x^2$.

**Answer:** $\frac{x^2}{2}-\frac{1}{2}\ln(1+x^2)+C$.

**(b)** Evaluate $I$ and show how the single substitution $u=1+x^2$ gives the same result. [3 marks]

Under the substitution, $x^3dx=\frac{(u-1)du}{2}$ and the limits are 1 and 2.

$I=\frac{1}{2}\int_1^2\left(1-\frac{1}{u}\right)du=\frac{1-\ln 2}{2}$.

**Answer:** $I=\frac{1-\ln 2}{2}$.

**3.** Consider $I = \int_0^{\frac{\pi}{2}} \cos^4 x \, dx$. [5 marks]

**(a)** Derive an identity expressing cosine to the fourth as a sum of cosines. [3 marks]

Square $\cos^2 x = \frac{1+\cos 2x}{2}$.

Reduce $\cos^2(2x)$ using $\frac{1+\cos 4x}{2}$ and collect terms.

**Answer:** $\cos^4 x = \frac{3+4\cos 2x+\cos 4x}{8}$.

**(b)** Evaluate $I$ exactly. [2 marks]

Integrate to $\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$.

Both sine terms vanish at both endpoints.

**Answer:** $\frac{3\pi}{16}$.

**4.** Consider $f(x) = \frac{x^2+2}{x^3-x}$. [8 marks]

**(a)** Find its partial-fraction decomposition. [3 marks]

Factor the denominator as $x(x-1)(x+1)$ and clear denominators.

At $x = 0$ the coefficient of $\frac{1}{x}$ is $-2$; at $x = 1$ and $-1$ the other coefficients are both $\frac{3}{2}$.

**Answer:** $f(x) = -\frac{2}{x} + \frac{3}{2(x-1)} + \frac{3}{2(x+1)}$.

**(b)** Evaluate its integral from 2 to 3 exactly. [3 marks]

An antiderivative on this interval is $-2\ln x + \frac{3}{2}\ln(x-1) + \frac{3}{2}\ln(x+1)$.

Combine endpoint differences in the two latter logs to obtain $\frac{3}{2}\ln\left[\frac{3^2-1}{2^2-1}\right]$.

**Answer:** $-2\ln(\frac{3}{2}) + \frac{3}{2}\ln(\frac{8}{3})$.

**(c)** Explain why the same ordinary endpoint calculation cannot be used from 1 to 3. [2 marks]

The integrand is undefined there; the ordinary continuity hypothesis fails and its logarithmic antiderivative has no finite endpoint value.

**Answer:** There is a pole at the lower endpoint $x = 1$.

**5.** Seek an antiderivative of $e^{2x}\cos x$ in the form $F(x) = e^{2x}(A\cos x + B\sin x)$. [6 marks]

**(a)** Differentiate the proposed form and find $A$ and $B$. [3 marks]

The derivative is $e^{2x}[(2A + B)\cos x + (2B - A)\sin x]$.

Solve $2A + B = 1$, $2B - A = 0$.

**Answer:** $A = \frac{2}{5}, B = \frac{1}{5}$.

**(b)** Evaluate $\int_0^{\pi} e^{2x}\cos x\,dx$. [3 marks]

The antiderivative values are $-\frac{2e^{2\pi}}{5}$ at $\pi$ and $\frac{2}{5}$ at zero.

Subtract the complete lower value.

**Answer:** $-\frac{2(e^{2\pi}+1)}{5}$.

**6.** Let $I = \int_0^{\ln 2} \frac{e^x}{\sqrt{1+e^x}}\,dx$. [5 marks]

**(a)** Choose a substitution, change the bounds and evaluate $I$. [3 marks]

Let $u = 1 + e^x$, so $du = e^x\,dx$ and the bounds become 2 and 3.

$I = \int_2^3 u^{-\frac{1}{2}}\,du = [2\sqrt{u}]_2^3$.

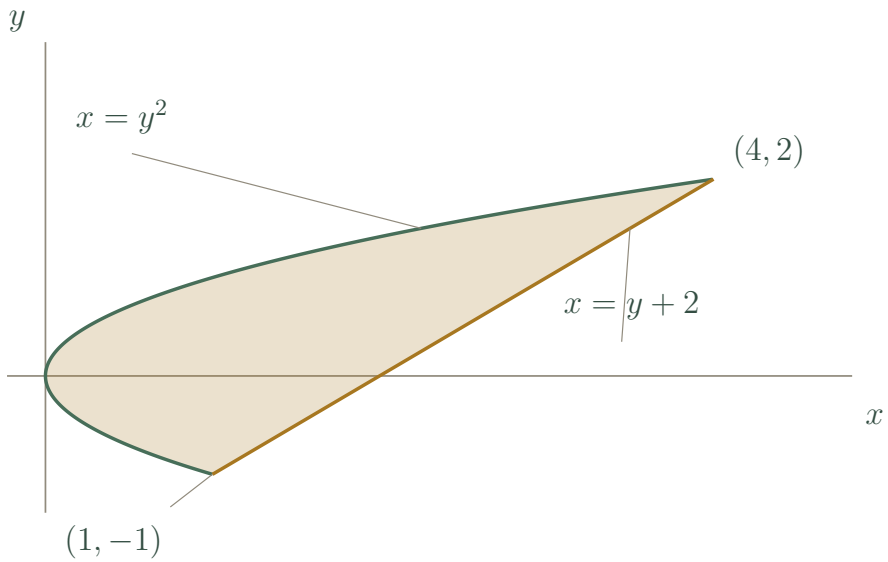
**Answer:** $I = 2(\sqrt{3} - \sqrt{2})$.

**(b)** Find the average value of the integrand on the original interval. [2 marks]

Divide the integral by the positive interval length $\ln 2$.

**Answer:** $\frac{2(\sqrt{3}-\sqrt{2})}{\ln 2}$.

**7.** The region bounded by $x = y^2$ and $x = y + 2$ is rotated about the $y$ axis. [7 marks]




**(a)** Find the two intersection $y$ values and the plane area. [3 marks]

$y^2 = y + 2$ gives $(y - 2)(y + 1) = 0$. Between the roots, $y + 2$ is the right boundary.

$A = \int_{-1}^{2} (y + 2 - y^2)\, dy = \frac{9}{2}$.

**Answer:** $y = -1, 2$; area $\frac{9}{2}$ square units.

**(b)** Find the exact solid volume using washers. [4 marks]

Both radii are non-negative on this interval: outer $y + 2$ and inner $y^2$.

$V = \pi \int_{-1}^{2} [(y + 2)^2 - y^4]\, dy = \frac{72\pi}{5}$.

**Answer:** $\frac{72\pi}{5}$ cubic units.

**8.** The region $0 \le x \le \pi$, $0 \le y \le \sin x$ is rotated about the $x$ axis. [6 marks]

**(a)** Find the total volume. [3 marks]

The disk radius is $\sin x$, so $V = \pi \int_0^{\pi} \sin^2 x \, dx$.

Use the half-angle identity; the integral is $\frac{\pi}{2}$.

**Answer:** $\frac{\pi^2}{2}$ cubic units.

**(b)** Find the volume from $x = 0$ to $a$, where $0 \le a \le \pi$, and show that the plane $x = \frac{\pi}{2}$ divides the solid into equal volumes. [3 marks]

Integrate the squared radius with upper limit $a$.

At $a = \frac{\pi}{2}$, the sine term is zero and the result is half the total volume.

**Answer:** $V(a) = \pi\left(\frac{a}{2} - \frac{\sin 2a}{4}\right)$; $V(\frac{\pi}{2}) = \frac{\pi^2}{4}$.

**9.** A tank starts with 30 L. During $0 \le t \le 4$ minutes, inflow is $6t(4 - t)$ L/min and outflow is a constant 8 L/min. [7 marks]

**(a)** Find the total inflow, total outflow and final amount. [3 marks]

$\int_0^4 (24t - 6t^2)\, dt = [12t^2 - 2t^3]_0^4 = 64$.

Outflow is $8(4) = 32$ L; add the net change 32 L to the initial 30 L.

**Answer:** Inflow $64$ L; outflow $32$ L; final amount $62$ L.

**(b)** Find an amount function $V(t)$, including its initial amount. [2 marks]

Integrate the net rate and impose $V(0) = 30$.

**Answer:** $V(t) = 30 + 12t^2 - 2t^3 - 8t$ L.

**(c)** Explain why the amount initially decreases even though the final amount exceeds 30 L. [2 marks]

The initial net rate is $-8$ L/min. Later positive accumulation can exceed this early loss.

**Answer:** At $t = 0$, the inflow is zero while the outflow is 8 L/min.

**10.** For $a > 0$, the region under $y = ax(1 - x)$ on $0 \le x \le 1$ has area $A$. Its rotation about the $x$ axis has volume $V$. [7 marks]

**(a)** Find $A$ and $V$ in terms of $a$. [4 marks]

$A = a\int_0^1 (x - x^2)\,dx = \frac{a}{6}$.

$V = \pi a^2 \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \pi a^2(\frac{1}{3} - \frac{1}{2} + \frac{1}{5})$.

**Answer:** $A = \frac{a}{6}$; $V = \frac{\pi a^2}{30}$.

**(b)** Eliminate $a$ to express $V$ in terms of $A$, then find $V$ when $A = 2$. [3 marks]

$a = 6A$, so substitute into the volume expression.

The quadratic dependence means doubling the area by vertical scaling quadruples this rotational volume.

**Answer:** $V = \frac{6\pi A^2}{5}$; for $A = 2$, $V = \frac{24\pi}{5}$.