Integration techniques and applications - practice test 5

10 questions | 58 marks | 75 minutes

Reasoning and applications across exact integrals, numerical approximation, changing regions and motion.

Equivalent correct methods earn the corresponding marks.

1. For all real x, $F'(x) = \frac{2x+3}{x^2+3x+5}$ and $F(0) = 0$. [5 marks]

(a) Find F and justify that it is defined on the whole real line. [3 marks]

Use the derivative-of-denominator pattern; the constant is $-\ln 5$.

$x^2 + 3x + 5 = (x + \frac{3}{2})^2 + \frac{11}{4} > 0$.

Answer: $F(x) = \ln\left(\frac{x^2+3x+5}{5}\right)$.

(b) Find $F(2) - F(0)$. [2 marks]

The logarithm ratio at 2 is $\frac{15}{5} = 3$.

Answer: $\ln 3$.

2. Let $I=\int_0^1 (1+x^2)^{-1}\,dx$ and $J=\int_0^1 (1+4x^2)^{-1}\,dx$. [5 marks]

(a) Evaluate I and J exactly. [3 marks]

For I use $\arctan x$. For J substitute $u=2x$, including $dx=\frac{du}{2}$.

Answer: $I=\frac{\pi}{4}$; $J=\frac{1}{2}\arctan 2$.

(b) Without using decimal approximations, explain which is larger. [2 marks]

For every $0<x\leq 1$, $1+x^2<1+4x^2$, so the first positive reciprocal is larger.

Integrating this strict inequality over a positive-length interval preserves it.

Answer: $I>J$.

3. For $x>0$, let $f(x)=\frac{x^2+x+1}{x(x+1)^2}$. [6 marks]

(a) Find A, B and C in $f=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$. [3 marks]

Clear denominators: $x^2+x+1=A(x+1)^2+Bx(x+1)+Cx$.

The constant gives $A=1$; then the quadratic gives $B=0$ and the linear gives $C=-1$.

Answer: $A=1, B=0, C=-1$.

(b) Evaluate the integral of f from 1 to 2. [3 marks]

An antiderivative is $\ln x+\frac{1}{x+1}$.

The endpoint difference is $\ln 2+\frac{1}{3}-\frac{1}{2}$.

Answer: $\ln 2-\frac{1}{6}$.

4. Let $I=\int_0^{\frac{\pi}{2}} e^x \sin x\,dx$ and $J=\int_0^{\frac{\pi}{2}} e^x \cos x\,dx$. [6 marks]

(a) Use integration by parts to show that $I+J$ equals $e^{\frac{\pi}{2}}$. [3 marks]

Choose $u=\sin x$, $dv=e^x\,dx$.

The endpoint product $[e^x \sin x]_0^{\frac{\pi}{2}}$ equals $e^{\frac{\pi}{2}}$.

Answer: $I=e^{\frac{\pi}{2}}-J$.

(b) Use parts on J, then solve for I. [3 marks]

With $u=\cos x$, $dv=e^x\,dx$, obtain $J=[e^x \cos x]_0^{\frac{\pi}{2}}+I=-1+I$.

Substitute into $I+J=e^{\frac{\pi}{2}}$ and solve $2I-1=e^{\frac{\pi}{2}}$.

Answer: $I=\frac{e^{\frac{\pi}{2}}+1}{2}$.

5. The curve is $y=x^3-x$ on $-2\le x\le 2$. [6 marks]

(a) Find all zeros in the interval and the signed integral. [2 marks]

Factor $x(x-1)(x+1)$.

The integrand is odd and the limits are symmetric.

Answer: Zeros $-1, 0, 1$; signed integral 0.

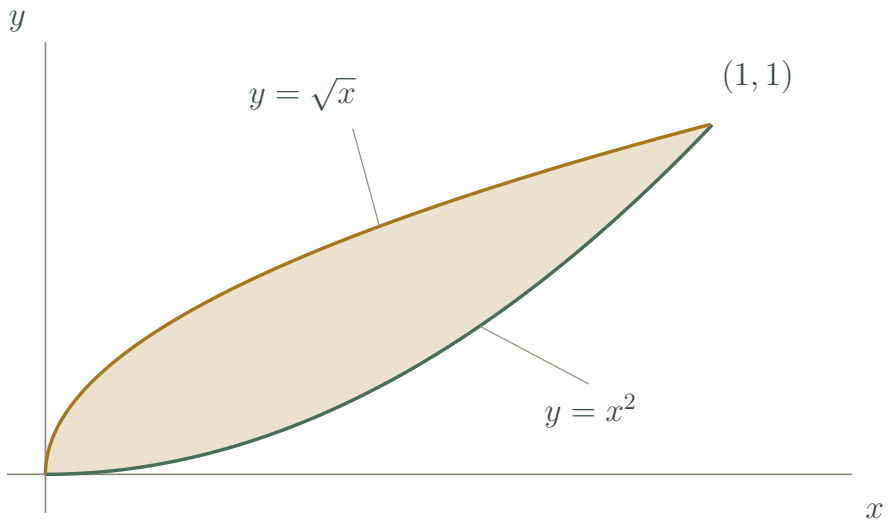
(b) Find the total geometric area between the curve and the axis. [4 marks]

Use symmetry and split the positive half at 1: $A=2[\int_0^1 (x-x^3)\,dx+\int_1^2 (x^3-x)\,dx]$.

The two bracketed areas are $\frac{1}{4}$ and $\frac{9}{4}$, so $A=5$.

Answer: 5 square units.

6. A region lies between $y=\sqrt{x}$ and $y=x^2$ in the first quadrant. [9 marks]



(a) Find the intersection points and the bounded area. [3 marks]

For $x \geq 0$, $\sqrt{x}=x^2$ gives $x=0$ or $x=1$.

On $(0, 1)$, the square root is above the square. Integrate $x^{\frac{1}{2}}-x^2$ to get $\frac{2}{3}-\frac{1}{3}$.

Answer: $(0, 0), (1, 1)$; area $\frac{1}{3}$ square unit.

(b) Find its volume when rotated about the x axis. [3 marks]

Outer radius is $\sqrt{x}$ and inner radius is x^2.

$V_x=\pi\int_0^1(x-x^4)\,dx=\pi(\frac{1}{2}-\frac{1}{5})$.

Answer: $\frac{3\pi}{10}$ cubic unit.

(c) Find its volume when rotated about the y axis, using horizontal slices. [3 marks]

The boundaries become $x=y^2$ and $x=\sqrt{y}$ for $0 \leq y \leq 1$.

$V_y=\pi\int_0^1(y-y^4)\,dy=\frac{3\pi}{10}$.

Answer: $\frac{3\pi}{10}$ cubic unit.

7. A tank has horizontal cross-sectional area $A(h) = \pi(4 - h^2)$ m^2 at height h metres, for $0 \le h \le 2$. [5 marks]

(a) Find the volume below height h and the full capacity. [3 marks]

Integrate the cross-sectional area from 0 to h.

Evaluate the resulting volume function at $h = 2$.

Answer: $V(h) = \pi\left(4h - \frac{h^3}{3}\right)$ m^3; capacity $\frac{16\pi}{3}$ m^3.

(b) At half the full height, what fraction of the full capacity is filled? [2 marks]

$V(1) = \frac{11\pi}{3}$.

Divide by $V(2) = \frac{16\pi}{3}$. Half height does not mean half volume because cross-sectional area varies.

Answer: $\frac{11}{16}$.

8. The table gives values of $f(x) = e^{-x^2}$. Use these values as supplied. [5 marks]

x	$f(x)$
0	1.0000
0.5	0.7788
1	0.3679

(a) Use the trapezium rule with two equal strips to estimate the integral from 0 to 1. [3 marks]

The strip width is $h = 0.5$.

$T = \frac{0.5}{2}[1 + 2(0.7788) + 0.3679] = 0.731375$.

Answer: 0.731375.

(b) A more accurate numerical value is 0.746824. Find the signed error T minus this value and the absolute error. [2 marks]

Subtract 0.746824 from the trapezium result.

The estimate is below the supplied more accurate value.

Answer: Signed error -0.015449; absolute error 0.015449.

9. A periodic temperature model is $T(t) = 3 + 2\cos t + \sin^2 t$ degrees C for $0 \le t \le 2\pi$ hours. [5 marks]

(a) Find its exact average temperature over this interval. [3 marks]

The constant contributes 6π, cosine integrates to zero, and sine squared contributes π.

Divide total 7π degree-hours by interval length 2π hours.

Answer: $\frac{7}{2}$ degrees C.

(b) Explain why the average is above 3 degrees C despite the cosine term averaging to zero. [2 marks]

A zero-mean cosine does not cancel a separate non-negative sine-squared contribution.

Answer: The sine-squared term has positive average $\frac{1}{2}$.

10. A particle moves on a straight line with velocity $v(t) = 3t^2 - 6t$ m/s for $0 \le t \le 3$. Its initial position is 2 m. [6 marks]

(a) Find its final position and average signed velocity. [3 marks]

The displacement is $[t^3 - 3t^2]_0^3 = 0$.

Add the initial position; divide displacement by 3 for average velocity.

Answer: Final position 2 m; average signed velocity 0 m/s.

(b) Find its total distance and average speed. [3 marks]

Velocity changes sign at $t = 2$. The signed changes over $[0, 2]$ and $[2, 3]$ are -4 m and 4 m.

Add their magnitudes, then divide by elapsed time.

Answer: Distance 8 m; average speed $\frac{8}{3}$ m/s.