Integration techniques and applications - practice test 4

10 questions | 54 marks | 75 minutes

Mixed techniques with translated inverse functions, repeated parts, horizontal regions and accumulated rates.

Equivalent correct methods earn the corresponding marks.

1. For all real x, let $f(x)=\frac{x+1}{\sqrt{x^2+2x+5}}$. [5 marks]

(a) Find an antiderivative using substitution. [3 marks]

Let $u=x^2+2x+5$, so $du=2(x+1)\,dx$.

Then $\frac{1}{2}\int u^{-\frac{1}{2}}\,du=\sqrt{u}+C$.

Answer: $\sqrt{x^2+2x+5}+C$.

(b) Find the member F for which $F(0)=3$. [2 marks]

At zero the radical equals $\sqrt{5}$, so $C=3-\sqrt{5}$.

The radicand is $(x+1)^2+4>0$ for every real input.

Answer: $F(x)=\sqrt{x^2+2x+5}+3-\sqrt{5}$.

2. Consider $f(x)=\frac{1}{\sqrt{4-(x-1)^2}}$. [5 marks]

(a) State its largest real open interval containing 1 and find an antiderivative there. [3 marks]

The denominator is real and non-zero exactly when $|x-1|<2$.

Use the arcsine rule with the translated variable $x-1$.

Answer: $-1<x<3$; $\arcsin\left(\frac{x-1}{2}\right)+C$.

(b) Evaluate its integral from 1 to 2 exactly. [2 marks]

The inverse-sine arguments at the endpoints are 0 and $\frac{1}{2}$.

Use radians: $\arcsin(\frac{1}{2})-\arcsin 0=\frac{\pi}{6}$.

Answer: $\frac{\pi}{6}$.

3. Consider $\int \frac{3x^2+1}{x^3+x+2}\,dx$ for $x \geq 0$. [5 marks]

(a) Find an antiderivative and justify the logarithm domain. [3 marks]

The numerator is the derivative of the denominator, so set $u = x^3 + x + 2$.

For $x \geq 0$, the argument is at least 2, so it is positive.

Answer: $\ln(x^3 + x + 2) + C$.

(b) Evaluate the definite integral from 0 to 1. [2 marks]

The denominator values change from 2 to 4.

The integral is $\ln 4 - \ln 2 = \ln 2$.

Answer: $\ln 2$.

4. Let $f(x) = \frac{x^2+2x+3}{(x+1)(x+2)}$. [6 marks]

(a) Divide first, then give the partial fractions. [3 marks]

The denominator is $x^2 + 3x + 2$, leaving numerator $-x + 1$ after division.

Write $-x + 1 = A(x+2) + B(x+1)$; $A + B = -1$ and $2A + B = 1$ give $A = 2, B = -3$.

Answer: $f(x) = 1 + \frac{2}{x+1} - \frac{3}{x+2}$.

(b) Find an antiderivative and state the excluded inputs. [3 marks]

Integrate the constant and reciprocal terms.

The formula applies on any interval avoiding both denominator zeros.

Answer: $x + 2\ln|x+1| - 3\ln|x+2| + C$; $x \neq -1, -2$.

5. Consider $\int x^2 \cos x \, dx$. [6 marks]

(a) Use integration by parts twice to find an antiderivative. [4 marks]

First choose $u = x^2$, $dv = \cos x \, dx$, giving $x^2 \sin x - 2\int x \sin x \, dx$.

A second application gives $\int x \sin x \, dx = -x \cos x + \sin x$. Substitute and simplify.

Answer: $x^2 \sin x + 2x \cos x - 2 \sin x + C$.

(b) Evaluate the integral from 0 to π and explain why a negative answer is possible. [2 marks]

Only $2x \cos x$ contributes at the upper endpoint, giving -2π.

The integral is signed accumulation; cosine is negative over the latter part of the interval.

Answer: -2π.

6. Let $I = \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx$. [4 marks]

(a) Derive a power-reduction expression for the integrand. [2 marks]

Square $\sin x \cos x = \frac{\sin 2x}{2}$.

Apply $\sin^2 (2x) = \frac{1 - \cos 4x}{2}$.

Answer: $\sin^2 x \cos^2 x = \frac{1 - \cos 4x}{8}$.

(b) Evaluate I exactly. [2 marks]

An antiderivative is $\frac{x}{8} - \frac{\sin 4x}{32}$.

The sine endpoint terms vanish, leaving $\frac{\pi}{8}$.

Answer: $\frac{\pi}{16}$.

7. For $x > 0$, consider $\int \ln(2x)\,dx$. [6 marks]

(a) Find an antiderivative using parts. [3 marks]

Choose $u = \ln(2x)$, $dv = dx$. Then $du = \frac{dx}{x}$ and $v = x$.

Subtract $\int 1\,dx$.

Answer: $x\ln(2x) - x + C$.

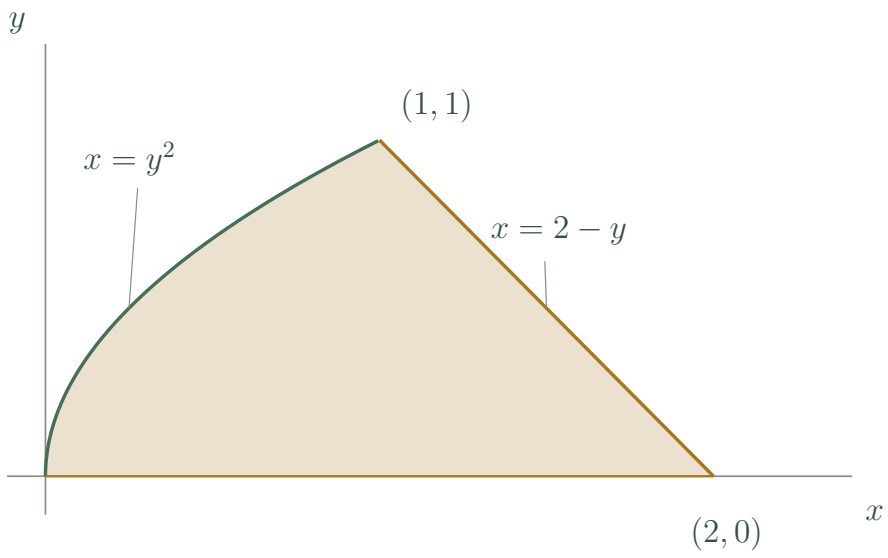
(b) Find the average value of $\ln(2x)$ on $[\frac{1}{2}, \frac{e}{2}]$. [3 marks]

The integral is $0 - (-\frac{1}{2}) = \frac{1}{2}$.

Divide by interval length $\frac{e-1}{2}$.

Answer: $\frac{1}{e-1}$.

8. A region is defined by $0 \le y \le 1$ and $y^2 \le x \le 2 - y$. [6 marks]



(a) Find its area using horizontal slices. [3 marks]

The right boundary is $2 - y$ and the left boundary is y^2.

$A = \int_0^1 (2 - y - y^2)\,dy = 2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$.

Answer: $\frac{7}{6}$ square units.

(b) Find the volume when the region is rotated about the y axis. [3 marks]

Use outer radius $2 - y$ and inner radius y^2.

$V = \pi\int_0^1 [(2 - y)^2 - y^4]\,dy = \pi(4 - 2 + \frac{1}{3} - \frac{1}{5})$.

Answer: $\frac{32\pi}{15}$ cubic units.

9. Two force laws along $0 \le x \le 2$ metres are $F_1(x) = 3x + 1$ N and $F_2(x) = 3x^2$ N. [5 marks]

(a) Find the work done by each force over the full displacement. [3 marks]

$W_1 = \left[\frac{3x^2}{2} + x\right]_0^2 = 6 + 2 = 8.$

$W_2 = [x^3]_0^2 = 8.$

Answer: Both do $8\ J$ of work.

(b) Does equal total work imply the forces are equal at every position? Use $x = 1$ to justify your answer. [2 marks]

An integral gives a total, so different profiles can share the same accumulated work.

The values at one common position already disprove equality of the force laws.

Answer: No: $F_1(1) = 4$ N and $F_2(1) = 3$ N.

10. A tank starts with 30 L. Water enters at $q(t) = 20te^{-t^2}$ L/min for $t \ge 0$, with no outflow. [6 marks]

(a) Find the amount added by time T. [3 marks]

Use $u = -t^2$ or recognise the derivative of e^{-t^2}.

$\int_0^T 20te^{-t^2}\,dt = [-10e^{-t^2}]_0^T.$

Answer: $10(1 - e^{-T^2})$ L.

(b) Find when 8 L has been added, and state the tank amount then. [3 marks]

$10(1 - e^{-T^2}) = 8$ gives $e^{-T^2} = \frac{1}{5}$.

Use $T \ge 0$ and add the initial 30 L.

Answer: $T = \sqrt{\ln 5}$ min; 38 L.