Integration techniques and applications - practice test 3

10 questions | 44 marks | 75 minutes

Diagnose reasoning and connect integration to models.

Equivalent correct methods earn the corresponding marks.

1. Two students give $F = \frac{1}{2}\ln(x^2+3)$ and $G = \ln\sqrt{x^2+3}$ as antiderivatives of $\frac{x}{x^2+3}$. [4 marks]

(a) Show that both answers are valid for every real x. [2 marks]

The log argument is positive.

Use $\ln(\sqrt{u}) = \frac{\ln u}{2}$ or differentiate directly.

Answer: They are equal, and each derivative is $\frac{x}{x^2+3}$.

(b) Explain what changes if arbitrary constants are included. [2 marks]

Their derivatives still agree.

An initial value would select one particular member.

Answer: The two selected antiderivatives may differ by a constant.

2. Consider $\int \frac{3e^{3x}}{1+e^{3x}}\,dx$. [4 marks]

(a) Find a suitable substitution and antiderivative. [2 marks]

The numerator is the derivative of the denominator.

The transformed integral is $\frac{du}{u}$.

Answer: $u = 1 + e^{3x}$; $\ln(e^{3x}+1) + C$.

(b) Evaluate it from 0 to $\frac{\ln 2}{3}$. [2 marks]

The transformed limits are 2 and 3.

Subtract $\ln 2$ from $\ln 3$.

Answer: $\ln\left(\frac{3}{2}\right)$.

3. Let $F'(x)=\frac{4x^5+3}{x^2}$ for $x>0$. [4 marks]

(a) Find the general antiderivative. [2 marks]

Use the power rule term by term.

Keep the reciprocal term on its positive-domain interval.

Answer: $\frac{x^5-3}{x}+C$.

(b) Find F when $F(1)=5$. [2 marks]

At $x=1$, the nonconstant terms total -2.

Thus $C=7$.

Answer: $x^4+7-\frac{3}{x}$.

4. Consider $\int\sin^2(3x)\,dx$. [4 marks]

(a) Rewrite the integrand and integrate. [2 marks]

Use $\sin^2(3x)=\frac{1-\cos(6x)}{2}$.

Account for the cosine frequency in the antiderivative.

Answer: $\frac{x}{2}-\frac{\sin(6x)}{12}+C$.

(b) Evaluate it from 0 to $\frac{\pi}{6}$. [2 marks]

The sine endpoint term is zero at both bounds.

The remaining term is half the interval length.

Answer: $\frac{\pi}{12}$.

5. For $x\geq 0$, integrate $\frac{6x+22}{(x+3)(x+4)}$. [5 marks]

(a) Find the partial fractions. [2 marks]

Clear denominators and compare coefficients.

The coefficients are $A=4, B=2$.

Answer: $\frac{4}{x+3}+\frac{2}{x+4}$.

(b) Find the general antiderivative, then its integral from 0 to 1. [3 marks]

The logarithm arguments are positive on the stated interval.

Subtract the endpoint values.

Answer: $4\ln(x+3)+2\ln(x+4)+C$; definite value $\ln\left(\frac{400}{81}\right)$.

6. Find $\int xe^{3x}\,dx$. [5 marks]

(a) Use integration by parts, stating u and dv. [3 marks]

Choose $u = x$ and $dv = e^{3x}dx$, so $v = \frac{e^{3x}}{3}$.

Subtract the remaining exponential integral.

Answer: $\frac{(3x-1)e^{3x}}{9} + C$.

(b) Evaluate the integral from 0 to 1. [2 marks]

Substitute both endpoints in the antiderivative.

Check the result is positive.

Answer: $\frac{1}{9} + \frac{2e^3}{9}$.

7. A proposed calculation is $\int_{-3}^{3} x^{-2}dx = \left[-\frac{1}{x}\right]_{-3}^{3}$. [4 marks]

(a) Explain why ordinary endpoint subtraction is invalid. [2 marks]

The continuity requirement for the ordinary fundamental-theorem calculation fails.

Answer: The integrand is undefined at $x = 0$ inside the interval.

(b) Evaluate instead $\int_{1}^{4} x^{-2}dx$. [2 marks]

This interval contains no pole.

Use $-\frac{1}{x}$ and subtract its endpoint values.

Answer: $\frac{3}{4}$.

8. A temperature profile is $T(x) = 20 + 3\sin x$ degrees C on $0 \le x \le \pi$. [4 marks]

(a) Find its exact average value. [2 marks]

Integrate the constant and sine terms.

Divide by the interval length π.

Answer: $20 + \frac{6}{\pi}$ degrees C.

(b) Explain why the average is not 20 degrees C. [2 marks]

The average includes this positive contribution.

Answer: The sine term is positive inside the interval, so it adds positive area.

9. A force $F(x) = 3x^2 + 2$ N acts along displacement $0 \le x \le 3$ m. [4 marks]

(a) Find the work done. [2 marks]

Integrate force with respect to displacement.

The endpoint result is $9(3) + 6$.

Answer: $33\ J$.

(b) Find the average force over the displacement. [2 marks]

Divide the work by the length of the displacement interval.

This has force units.

Answer: 11 N.

10. The region under $y = ax$ for $0 \le x \le 1$ is rotated about the x axis, with a positive. [6 marks]

(a) Express its volume in terms of a. [2 marks]

Integrate the squared disk radius a^2x^2.

Answer: $\frac{\pi a^2}{3}$.

(b) Find a if the volume is 3π. [2 marks]

Equate the volume expressions and cancel $\frac{\pi}{3}$.

Use the positive condition.

Answer: $a = 3$.

(c) If a doubles, explain the effect on volume. [2 marks]

The formula depends on the square of a.

Answer: Volume quadruples.