Integration techniques and applications - practice test 2

10 questions | 46 marks | 75 minutes

Combine techniques and interpret geometric and practical results.

Equivalent correct methods earn the corresponding marks.

1. A model has $F'(x) = 4e^{2x}$. [4 marks]

(a) Find F if $F(0) = 5$. [2 marks]

Integrate the exponential and divide by its inner coefficient.

$F(0) = 2 + C = 5$, so $C = 3$.

Answer: $F = 2e^{2x} + 3$.

(b) Evaluate $\int_0^{\frac{\ln(2)}{2}} 4e^{2x}\,dx$. [2 marks]

Use the antiderivative without its arbitrary constant.

The endpoint difference is 2.

Answer: 2.

2. For $|x| < \frac{1}{2}$, consider $f(x) = \frac{2}{\sqrt{1-4x^2}}$. [4 marks]

(a) Find an antiderivative and explain the inner factor. [2 marks]

The derivative of the inner argument is 2.

It matches the numerator.

Answer: $\arcsin(2x) + C$.

(b) Evaluate $\int_0^{\frac{1}{4}} f(x)\,dx$. [2 marks]

The inverse-sine argument at the upper endpoint is $\frac{1}{2}$.

Use radians.

Answer: $\frac{\pi}{6}$.

3. For $x > 0$, consider $\frac{2x+5}{(x+2)^2}$. [4 marks]

(a) Decompose it using both powers of the repeated factor. [2 marks]

Write the numerator as $2(x+2)+1$.

Divide each term by the common squared denominator.

Answer: $\frac{2}{x+2} + \frac{1}{(x+2)^2}$.

(b) Integrate and verify by differentiation. [2 marks]

Use a logarithm for the reciprocal term and the power rule for the squared reciprocal.

Differentiation recovers the original expression.

Answer: $\frac{2(x+2)\ln(x+2)-1}{x+2} + C$.

4. For $x > 0$, consider $\int x^1 \ln x \, dx$. [5 marks]

(a) Find the general antiderivative. [3 marks]

Choose $u = \ln x$ and integrate the power as dv.

Simplify the remaining power integral.

Answer: $\frac{x^2(2\ln(x)-1)}{4} + C$.

(b) Evaluate it from $x = 1$ to $x = e$. [2 marks]

Use $\ln e = 1$ and $\ln 1 = 0$.

Subtract the complete lower endpoint value.

Answer: $\frac{1}{4} + \frac{e^2}{4}$.

5. Consider $\int \sin^3(x)\,dx$. [5 marks]

(a) Find an antiderivative using an identity and substitution. [3 marks]

Keep one sine factor and replace the other square with $1-\cos^2(x)$.

Set $u=\cos(x)$ and include the factor -1.

Answer: $\frac{(\cos(2x)-5)\cos(x)}{6}+C$.

(b) Evaluate it from 0 to $\frac{\pi}{2}$. [2 marks]

The endpoint cosine values are 1 and 0.

Substitute in the antiderivative.

Answer: $\frac{2}{3}$.

6. Consider $\int x\ln(x^2+1)\,dx$. [6 marks]

(a) Use substitution followed by parts to integrate. [4 marks]

Set $u=x^2+1$ so $x\,dx=\frac{du}{2}$.

Integrate $\ln u$ by parts.

Answer: $-\frac{x^2}{2}+\frac{(x^2+1)\ln(x^2+1)}{2}-\frac{1}{2}+C$.

(b) Evaluate the original integral from 0 to 1. [2 marks]

Use transformed limits 1 and 2.

Subtract complete endpoint expressions.

Answer: $-\frac{1}{2}+\ln(2)$.

7. A region is bounded by $y=3x-x^2$ and the x axis. [5 marks]

(a) Find its horizontal endpoints. [2 marks]

Factor the quadratic as $x(3-x)$.

Answer: $x=0$ and $x=3$.

(b) Find the exact region area. [3 marks]

The curve is above the axis between these roots.

Integrate it to obtain $\frac{9}{2}$.

Answer: $\frac{9}{2}$ square units.

8. A region is $0 \le y \le 2$ and $0 \le x \le y^2$. [5 marks]

(a) Find its plane area. [2 marks]

Use horizontal strips of width y^2.

Answer: $\frac{8}{3}$ square units.

(b) Find its volume about the y axis. [3 marks]

The disk radius is y^2, so the squared radius is y^4.

Integrate πy^4 from 0 to 2.

Answer: $\frac{32\pi}{5}$ cubic units.

9. A tank starts with 20 L. Water enters at $q(t) = 2t + 2$ L/min for $0 \le t \le 3$. [4 marks]

(a) Find the added water and the final amount. [2 marks]

Integrate q over the three-minute interval.

Add the initial amount only after calculating the change.

Answer: Added 15 L; final 35 L.

(b) Find the average inflow rate. [2 marks]

Divide accumulated inflow by elapsed time.

The units remain litres per minute.

Answer: 5 L/min.

10. A student claims $G(x) = xe^{2x}$ is an antiderivative of e^{2x}. [4 marks]

(a) Differentiate G and identify the mistake. [2 marks]

Use the product rule.

The extra product term cannot be discarded.

Answer: $G' = e^{2x}(1 + 2x)$, so the claim is false.

(b) Give a correct antiderivative of xe^{2x}. [2 marks]

Apply integration by parts.

Verify the result by differentiation.

Answer: $e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right) + C$.