Integration techniques and applications - practice test 1

10 questions | 44 marks | 75 minutes

Select methods and establish reliable exact working.

Equivalent correct methods earn the corresponding marks.

1. Let $F'(x) = 2x + \frac{3}{x^2}$ for $x > 0$. [4 marks]

(a) Find the general antiderivative. [2 marks]

Use the power rule term by term.

Keep the reciprocal term on its positive-domain interval.

Answer: $\frac{x^3 - 3}{x} + C$.

(b) Find F when $F(1) = 5$. [2 marks]

At $x = 1$, the nonconstant terms total -2.

Thus $C = 7$.

Answer: $x^2 + 7 - \frac{3}{x}$.

2. Consider $f(x) = \frac{1}{4 + x^2}$. [4 marks]

(a) Find an antiderivative. [2 marks]

Recognise the sum-of-squares denominator.

Use $a = 2$ in the arctangent formula.

Answer: $\frac{\operatorname{atan}\left(\frac{x}{2}\right)}{2} + C$.

(b) Find the exact integral from $x = 0$ to $x = 2$. [2 marks]

At the upper endpoint, $\arctan(1) = \frac{\pi}{4}$.

The lower endpoint contributes zero.

Answer: $\frac{\pi}{8}$.

3. Consider $\int x(1+x^2)^2\,dx$. [5 marks]

(a) Use a substitution to find the general antiderivative. [3 marks]

Set $u = 1 + x^2$ and $du = 2x\,dx$.

Integrate $\frac{u^2}{2}$ and return to x.

Answer: $\frac{(x^2+1)^3}{6} + C$.

(b) Evaluate the integral from 0 to 1 using transformed limits. [2 marks]

The new limits are $u = 1$ and $u = 2$.

Endpoint subtraction gives $\frac{7}{6}$.

Answer: $\frac{7}{6}$.

4. Consider $\int \sin^2(x)\,dx$. [4 marks]

(a) Rewrite the integrand and integrate. [2 marks]

Use $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

Account for the cosine frequency in the antiderivative.

Answer: $\frac{x}{2} - \frac{\sin(2x)}{4} + C$.

(b) Evaluate it from 0 to $\frac{\pi}{2}$. [2 marks]

The sine endpoint term is zero at both bounds.

The remaining term is half the interval length.

Answer: $\frac{\pi}{4}$.

5. For $x \geq 0$, integrate $\frac{5x+13}{(x+2)(x+3)}$. [5 marks]

(a) Find the partial fractions. [2 marks]

Clear denominators and compare coefficients.

The coefficients are $A = 3, B = 2$.

Answer: $\frac{3}{x+2} + \frac{2}{x+3}$.

(b) Find the general antiderivative, then its integral from 0 to 1. [3 marks]

The logarithm arguments are positive on the stated interval.

Subtract the endpoint values.

Answer: $3\ln(x+2) + 2\ln(x+3) + C$; definite value $\ln(6)$.

6. Find $\int xe^x\,dx$. [5 marks]

(a) Use integration by parts, stating u and dv. [3 marks]

Choose $u = x$ and $dv = e^x dx$, so $v = e^x$.

Subtract the remaining exponential integral.

Answer: $(x-1)e^x + C$.

(b) Evaluate the integral from 0 to 1. [2 marks]

Substitute both endpoints in the antiderivative.

Check the result is positive.

Answer: 1.

7. Consider $\int \frac{1}{x^2+2x+5}\,dx$. [4 marks]

(a) Complete the square and find an antiderivative. [2 marks]

The denominator is $(x+1)^2+4$.

Apply the shifted arctangent pattern.

Answer: $\frac{\operatorname{atan}\left(\frac{x}{2}+\frac{1}{2}\right)}{2} + C$.

(b) Evaluate it from $x = -1$ to $x = 1$. [2 marks]

The arctangent arguments run from 0 to 1.

Multiply $\frac{\pi}{4}$ by the outside factor $\frac{1}{2}$.

Answer: $\frac{\pi}{8}$.

8. The curve is $y = x(x-2)$ on $0 \le x \le 2$. [4 marks]

(a) Evaluate its signed integral over the interval. [2 marks]

The antiderivative is $\frac{x^3}{3} - \frac{2x^2}{2}$.

Subtract endpoints.

Answer: $-\frac{4}{3}$.

(b) Find the geometric area and explain the sign difference. [2 marks]

The curve lies below the x axis between its zeros.

Area is the negative of this single below-axis integral.

Answer: $\frac{4}{3}$ square units.

9. Consider $y = 2x$ and $y = x^2$ in the first quadrant. [4 marks]

(a) Find both intersections and identify the upper curve between them. [2 marks]

Solve $x^2 = 2x$.

A positive midpoint shows the line is above the parabola.

Answer: $(0, 0)$ and $(2, 4)$; the line is upper.

(b) Find the bounded area. [2 marks]

Integrate line minus parabola between the roots.

The result is $\frac{4}{3}$.

Answer: $\frac{4}{3}$ square units.

10. The triangular region $0 \le x \le 2$, $0 \le y \le x$ is rotated. [5 marks]

(a) Find its volume about the x axis. [2 marks]

The disk radius is x.

Use $\pi \int_0^2 x^2 dx$.

Answer: $\frac{8\pi}{3}$ cubic units.

(b) Find its volume about the y axis using horizontal washers. [3 marks]

At height y, the radii are 2 and y.

Use $\pi \int_0^2 (4 - y^2) dy$.

Answer: $\frac{16\pi}{3}$ cubic units.