Vectors in three dimensions - practice test 6

10 questions | 73 marks | 75 minutes

Interpret data, diagnose errors and solve connected spatial problems, finishing with parameter cases.

Equivalent correct methods earn the corresponding marks.

1. A drone travels in straight segments from A to B, then B to C. Coordinates are in metres. [6 marks]

Point	x (m)	y (m)	z (m)
A	0	0	0
B	3	4	0
C	3	4	12

(a) Find the total path length. [2 marks]

$AB = (3, 4, 0)$ has length 5 m.

$BC = (0, 0, 12)$ has length 12 m; add the two segment lengths.

Answer: $17\ \text{m}$.

(b) Find the displacement from A to C and its magnitude. [2 marks]

Subtract A from C.

The squared displacement length is $9 + 16 + 144 = 169$.

Answer: $(3, 4, 12)\ \text{m}$, magnitude $13\ \text{m}$.

(c) Give a unit vector in the direction from A to C. [2 marks]

Divide the displacement vector by its magnitude 13.

Answer: $(\frac{3}{13}, \frac{4}{13}, \frac{12}{13})$.

2. A vector has the form $\mathbf{a} = (1, -1, 2) + \lambda(1, 1, 0)$, where λ is real, and $|\mathbf{a}| = \sqrt{14}$. [7 marks]

(a) Verify that the two displayed component directions are perpendicular. [2 marks]

Compute $1(1) + (-1)(1) + 2(0) = 0$.

Answer: Their dot product is zero.

(b) Find every possible value of λ. [3 marks]

The squared length is $(1 + \lambda)^2 + (-1 + \lambda)^2 + 4 = 6 + 2\lambda^2$.

Set this equal to 14 to obtain $\lambda^2 = 4$.

Answer: $\lambda = 2$ or $\lambda = -2$.

(c) If $\mathbf{a} \cdot (1, 1, 0) > 0$, find a. [2 marks]

The dot product is 2λ, so choose the positive root 2.

Substitute it in the original expression.

Answer: $\mathbf{a} = (3, 1, 2)$.

3. For $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (2, -1, 3)$, a student claims $\mathbf{u} \times \mathbf{v} = (5, 5, -5)$. [7 marks]

(a) Use a dot-product check to show that the claimed cross product is incorrect. [2 marks]

Compute $(5, 5, -5) \cdot (1, 2, -1) = 5 + 10 + 5 = 20$.

Answer: Its dot product with u is 20, so it is not perpendicular to u.

(b) Find the correct cross product and check both perpendicularities. [3 marks]

The components are $(2(3) - (-1)(-1), (-1)(2) - 1(3), 1(-1) - 2(2))$.

The two checks give $5 - 10 + 5 = 0$ and $10 + 5 - 15 = 0$.

Answer: $(5, -5, -5)$.

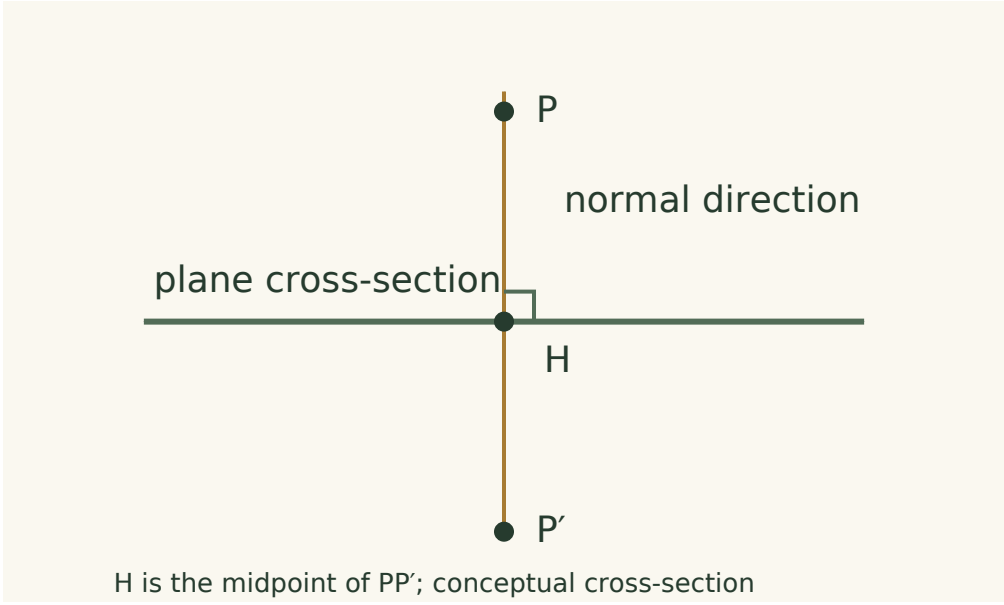
(c) Find the area of the triangle spanned by u and v. [2 marks]

The cross product has length $\sqrt{25 + 25 + 25} = 5\sqrt{3}$.

Halve this for a triangle.

Answer: $\frac{5\sqrt{3}}{2}$ square units.

4. Point $P=(3,0,0)$ is reflected in the plane $x-y+z=2$. The reflection P' is defined so that the perpendicular foot H is the midpoint of PP'. [7 marks]



H is the midpoint of PP′; conceptual cross-section

(a) Find H. [3 marks]

A normal is $n=(1,-1,1)$. Write $H=P-\mu n$.

Substitute in the plane: $3-3\mu=2$, so $\mu=\frac{1}{3}$.

Answer: $H=(\frac{8}{3},\frac{1}{3},-\frac{1}{3})$.

(b) Find the reflection P'. [2 marks]

The midpoint relation gives $P'=2H-P$.

Combine the coordinates.

Answer: $P'=(\frac{7}{3},\frac{2}{3},-\frac{2}{3})$.

(c) Find the length PP' and explain why it is twice the distance of P from the plane. [2 marks]

PH has length $\frac{|3-2|}{\sqrt{3}}=\frac{1}{\sqrt{3}}$.

H is the midpoint and PP' follows the normal, so $PP'=2PH$.

Answer: $\frac{2}{\sqrt{3}}$ units.

5. Two robots move in a common coordinate system according to $P(t) = (t, 0, 0)$ and $Q(t) = (2, 3 - t, 0)$ for $0 \le t \le 5$, with t in seconds and coordinates in metres. [7 marks]

(a) Write their squared separation at time t. [2 marks]

Subtract the simultaneous position vectors.

Square and add the nonzero coordinate differences.

Answer: $D^2 = (t-2)^2 + (t-3)^2$.

(b) Find their minimum separation and the time when it occurs, without calculus. [3 marks]

Complete the square: $D^2 = 2(t - \frac{5}{2})^2 + \frac{1}{2}$.

The minimum occurs at $t = \frac{5}{2}$, which is inside the stated time interval.

Answer: At $t = \frac{5}{2}$ seconds, the separation is $\frac{1}{\sqrt{2}}$ metres.

(c) Do the robots collide? Justify. [2 marks]

A collision would require $t = 2$ from the x coordinates and $t = 3$ from the y coordinates at once.

Equivalently, their minimum separation is positive.

Answer: No.

6. A straight cable segment has endpoints $A(1,0,0)$ and $B(3,2,2)$. An infinite plane has equation $x+y+z=10$. [7 marks]

(a) Parametrise the cable segment, stating the parameter interval. [2 marks]

Use $B-A=(2,2,2)$ and restrict t to the endpoints.

Answer: $\mathbf{r}=(1,0,0)+t(2,2,2),\ 0\le t\le 1.$

(b) Find where the complete line through A and B meets the plane. [2 marks]

Substitution gives $1+6t=10$.

Solve for t and substitute in all three line components.

Answer: $t=\frac{3}{2}$, point $(4,3,3)$.

(c) Does the cable segment meet the plane? Find its closest endpoint and that endpoint's distance from the plane. [3 marks]

The intersection parameter is outside $[0,1]$. On this interval the plane expression rises from 1 to 7, remaining below 10.

Thus B is closest; its distance is $\frac{|7-10|}{\sqrt{3}}=\sqrt{3}$.

Answer: No. B is closest, at distance $\sqrt{3}$ units.

7. Two planes are $\Pi_1 : x + y + z = 4$ and $\Pi_2 : 2x - y + z = 2$. [7 marks]

(a) Find a direction vector for their intersection. [2 marks]

Cross the normals $(1, 1, 1)$ and $(2, -1, 1)$.

Answer: $(2, 1, -3)$.

(b) Give a vector equation for the intersection line. [3 marks]

Subtract the first equation from the second to get $x - 2y = -2$.

Let $y = t$, then $x = 2t - 2$ and $z = 6 - 3t$.

Answer: $\mathbf{r} = (-2, 0, 6) + t(2, 1, -3)$.

(c) Find the smaller angle between the planes to one decimal degree. [2 marks]

The normal dot product is 2, and the magnitudes are $\sqrt{3}$ and $\sqrt{6}$.

Thus $\cos\theta = \frac{2}{\sqrt{18}} = \frac{\sqrt{2}}{3}$; take inverse cosine.

Answer: 61.9°.

8. Two support lines are $\ell_1 : (1, 0, 0) + s(1, 1, 1)$ and $\ell_2 : (0, 2, 0) + t(1, -1, 0)$, for real s, t. [9 marks]

(a) Find a common perpendicular direction and the shortest line separation. [3 marks]

The cross product of the line directions is $(1, 1, -2)$.

The displacement of initial points is $(-1, 2, 0)$, whose dot product with this normal is 1. Divide its absolute value by $\sqrt{6}$.

Answer: A direction is $(1, 1, -2)$; distance $\frac{1}{\sqrt{6}}$ units.

(b) Find the closest points P on ℓ_1 and Q on ℓ_2. [4 marks]

The joining vector is $(-1 + t - s, 2 - t - s, -s)$.

Its dot products with the directions give $1 - 3s = 0$ and $-3 + 2t = 0$, so $s = \frac{1}{3}$ and $t = \frac{3}{2}$.

Answer: $P = (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$ and $Q = (\frac{3}{2}, \frac{1}{2}, 0)$.

(c) Verify the joining vector is perpendicular to both lines. [2 marks]

Dot with $(1, 1, 1)$ to get $\frac{1}{6} + \frac{1}{6} - \frac{1}{3} = 0$.

Dot with $(1, -1, 0)$ to get $\frac{1}{6} - \frac{1}{6} = 0$.

Answer: $\overrightarrow{PQ} = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$; both dot products are zero.

9. Consider the homogeneous system $x+y+z=0$, $2x-y+z=0$, $3x+2z=0$. [7 marks]

(a) Explain why the system cannot have no solution. [2 marks]

Substitute $x=y=z=0$ into all three equations.

Answer: The origin always satisfies every homogeneous linear equation.

(b) Find every solution and interpret the common geometry. [3 marks]

Equation 3 is the sum of equations 1 and 2.

Subtract equation 1 from equation 2 to get $x=2y$; set $y=t$, then $z=-3t$.

Answer: $\mathbf{r}=t(2,1,-3)$, real t; the three planes share a line.

(c) If the extra constraint $x=0$ is imposed, what remains? [2 marks]

Since $x=2t$, the extra condition gives $t=0$.

Answer: Only the origin $(0,0,0)$.

10. A system depends on real parameters a and b: $x+y+z=3$, $x+2y+3z=6$, $x+2y+az=b$. [9 marks]

(a) For $a\neq 3$, find the unique solution in terms of a,b. [4 marks]

Subtract equation 2 from equation 3 to get $(a-3)z=b-6$.

Subtract equation 1 from equation 2 to get $y+2z=3$; then the first equation gives $x=z$.

Answer: $z=\frac{b-6}{a-3}$, $x=z$, $y=3-2z$.

(b) Classify and parametrise the solution set when $a=3, b=6$. [3 marks]

Equations 2 and 3 coincide.

Choose $z=t$ and use the first two independent equations.

Answer: Infinitely many solutions $(t,3-2t,t)$, real t; a line.

(c) Classify the case $a=3$, $b\neq 6$ and justify it. [2 marks]

Equations 2 and 3 then have identical left sides and different constants, which is a contradiction.

Answer: No solution.