Vectors in three dimensions - practice test 5

10 questions | 60 marks | 70 minutes

Use parameters, proof and perpendicularity in connected tasks.

Equivalent correct methods earn the corresponding marks.

1. A constant force $\mathbf{F} = (2, 0, 1)$ N acts during displacement $\mathbf{d} = (-2, 1, 2)$ m. [7 marks]

(a) Calculate the work done. [2 marks]

Work is the dot product: $2(-2) + 0(1) + 1(2) = -2$.

Answer: -2 J.

(b) Find the vector component of F along the displacement direction. [3 marks]

The projection coefficient is $\frac{\mathbf{F} \cdot \mathbf{d}}{\mathbf{d} \cdot \mathbf{d}} = -\frac{2}{9}$.

Multiply d by $-\frac{2}{9}$.

Answer: $(\frac{4}{9}, -\frac{2}{9}, -\frac{4}{9})$ N.

(c) Explain the sign of the work in relation to the projected force. [2 marks]

The projection coefficient is negative; this indicates opposition to the chosen displacement direction.

Answer: The projected force points against the displacement, so the work is negative.

2. Let $\mathbf{a}, \mathbf{b}$ be real three-dimensional vectors with $\mathbf{b} \neq \mathbf{0}$. Define $\mathbf{p} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$ and $\mathbf{q} = \mathbf{a} - \mathbf{p}$. [7 marks]

(a) Prove that $\mathbf{q} \cdot \mathbf{b} = 0$. [3 marks]

Since $\mathbf{b} \neq \mathbf{0}$, $|\mathbf{b}|^2 > 0$, so the projection coefficient is defined. [1 mark]

$\mathbf{q} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}(\mathbf{b} \cdot \mathbf{b})$. [1 mark]

Using $\mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|^2$, the two terms cancel and give zero. [1 mark]

Answer: $\mathbf{q} \cdot \mathbf{b} = 0$.

(b) Hence prove that $|\mathbf{a}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2$. [2 marks]

The vector $\mathbf{p}$ is a scalar multiple of $\mathbf{b}$, so part (a) gives $\mathbf{p} \cdot \mathbf{q} = 0$. [1 mark]

$|\mathbf{a}|^2 = |\mathbf{p} + \mathbf{q}|^2 = |\mathbf{p}|^2 + 2\mathbf{p} \cdot \mathbf{q} + |\mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2$. [1 mark]

Answer: $|\mathbf{a}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2$.

(c) For $\mathbf{a} = (5, 1, 2)$ and $\mathbf{b} = (1, 0, 1)$, calculate $\mathbf{p}$ and $\mathbf{q}$ and check the squared-length equality. [2 marks]

$\mathbf{a} \cdot \mathbf{b} = 7$ and $|\mathbf{b}|^2 = 2$, giving $\mathbf{p} = (\frac{7}{2}, 0, \frac{7}{2})$ and $\mathbf{q} = (\frac{3}{2}, 1, -\frac{3}{2})$. [1 mark]

$|\mathbf{a}|^2 = 25 + 1 + 4 = 30$, while $|\mathbf{p}|^2 + |\mathbf{q}|^2 = \frac{49}{2} + \frac{11}{2} = 30$. [1 mark]

Answer: $\mathbf{p} = (\frac{7}{2}, 0, \frac{7}{2})$, $\mathbf{q} = (\frac{3}{2}, 1, -\frac{3}{2})$; $30 = \frac{49}{2} + \frac{11}{2}$.

3. A parallelogram $ABCD$ has consecutive vertices $A(1, 0, 2)$, $B(3, 1, 1)$, $C(4, 4, 3)$. [6 marks]

(a) Find D. [2 marks]

Opposite side vectors agree, so $\mathbf{d} = \mathbf{a} + \mathbf{c} - \mathbf{b}$.

Combine the three coordinates.

Answer: $D = (2, 3, 4)$.

(b) Verify that its diagonals bisect one another. [2 marks]

Average A and C, then average B and D.

The two midpoint vectors agree.

Answer: Both diagonal midpoints are $(\frac{5}{2}, 2, \frac{5}{2})$.

(c) Is this parallelogram a rhombus? Justify. [2 marks]

The adjacent side vectors are $AB = (2, 1, -1)$ and $AD = (1, 3, 2)$.

Their squared lengths are 6 and 14, so the adjacent sides have different lengths.

Answer: No.

4. Let $\mathbf{a} = (5, 2, 3)$ and $\mathbf{b} = (1, 1, 0)$. [5 marks]

(a) Find the vector projection of a along b. [2 marks]

Projection coefficient is $\frac{7}{2}$.

Multiply this scalar by b.

Answer: $(\frac{7}{2}, \frac{7}{2}, 0)$

(b) Find and verify the perpendicular component. [3 marks]

Subtract the projection from a. Dotting $(\frac{3}{2}, -\frac{3}{2}, 3)$ with b gives 0.

Answer: $(\frac{3}{2}, -\frac{3}{2}, 3)$

5. Two planes have equations $\Pi_1 : x + 2y - z = 4$ and $\Pi_2 : 2x + 4y - 2z = k$. [7 marks]

(a) Explain why their normals are parallel. [2 marks]

The coefficient vectors are $(1, 2, -1)$ and $(2, 4, -2)$.

Answer: The second normal is twice the first.

(b) For which k are the planes coincident? Describe all remaining cases. [2 marks]

Multiply every term of the first equation by 2.

Its constant becomes 8, so equality of both complete equations requires $k = 8$.

Answer: Coincident for $k = 8$; distinct parallel planes otherwise.

(c) For $k = 10$, find the plane separation. [3 marks]

Divide the second equation by 2 to get $x + 2y - z = 5$.

The equal-normal constants differ by 1; divide by the normal magnitude $\sqrt{6}$.

Answer: $\frac{1}{\sqrt{6}}$ units.

6. Consider $\ell : \mathbf{r} = (0, 0, 5) + t(1, -1, a)$ and plane $x + y + z = 3$. [4 marks]

(a) For a nonzero, find the parameter of the intersection. [2 marks]

Substitution gives $5 + at = 3$.

Divide by the stated nonzero a.

Answer: $t = -\frac{2}{a}$.

(b) Classify the case $a = 0$. [2 marks]

With $a = 0$, the left side of the plane equation stays 5.

Compare this constant with 3.

Answer: The line is parallel to and disjoint from the plane.

7. $P=(0,5,2)$ and $\ell:\mathbf{r}=(1,0,0)+t(1,1,0)$. [5 marks]

(a) Find the parameter of the closest point Q. [2 marks]

Perpendicularity gives $(1+t,t-5,-2)\cdot(1,1,0)=0$.

This simplifies to $2t+1-5=0$.

Answer: $t=2$

(b) Find Q and the shortest distance. [3 marks]

Substitute the parameter into the line.

Calculate $|Q-P|=\sqrt{22}$.

Answer: $Q=(3,2,0)$, distance $\sqrt{22}$.

8. Consider lines $\ell_1=s(1,1,0)$ and $\ell_2=(0,5,1)+t(1,0,1)$. [7 marks]

(a) Show that the lines are skew. [2 marks]

An intersection would require $t=-1$ from the z coordinate and $s=-1$ from the x coordinate.

Then the y coordinate would require $-1=5$, which is impossible.

Answer: Their directions are not parallel and the lines do not intersect.

(b) Find the closest points P and Q, and their distance. [5 marks]

The joining vector is $(t-s,5-s,1+t)$. Perpendicularity gives $t+5-2s=0$ and $2t-s+1=0$.

Solve to obtain $s=3$ and $t=1$. Substitute in the two lines and calculate the joining-vector magnitude.

Answer: $P=(3,3,0)$, $Q=(1,5,2)$; distance $2\sqrt{3}$.

9. An augmented system has rows $[1, 1, 1 \mid 6]$, $[2, 3, 1 \mid 13]$, $[1, 0, 2 \mid 5]$. A student applies $R_2 \leftarrow R_2 - 2R_1$ but writes $[0, 1, -1 \mid 13]$. [7 marks]

(a) Identify the error and write the correct new row 2. [2 marks]

The augmented constant becomes $13 - 2(6) = 1$, not 13.

Answer: The constant must also change; the row is $[0, 1, -1 \mid 1]$.

(b) Apply $R_3 \leftarrow R_3 - R_1$ and compare the last two rows. [2 marks]

Subtract every entry of row 1 from row 3.

The two remaining equations are dependent.

Answer: $R_3 = [0, -1, 1 \mid -1]$, the negative of the corrected row 2.

(c) Describe the entire solution set and its geometry. [3 marks]

Let $z = t$; row 2 gives $y = 1 + t$.

Row 1 gives $x = 6 - y - z = 5 - 2t$. There are two pivots and one free parameter.

Answer: $(x, y, z) = (5 - 2t, 1 + t, t)$ for real t; a line.

10. An augmented system has rows $[1, 2, -1 \mid 4]$, $[3, 7, -2 \mid 13]$, $[0, 0, 1 \mid 5]$. [5 marks]

(a) Apply $R_2 \leftarrow R_2 - 3R_1$. [2 marks]

Subtract 3 times every entry of row 1, including the constant.

Answer: The new row 2 is $[0, 1, 1 \mid 1]$.

(b) Find the unique solution. [3 marks]

Row 3 gives $z = 5$; row 2 gives $y = -4$.

Row 1 gives $x = 4 - 2y + z = 17$.

Answer: $(x, y, z) = (17, -4, 5)$.