Vectors in three dimensions - practice test 4

10 questions | 51 marks | 70 minutes

Solve mixed geometric and algebraic problems.

Equivalent correct methods earn the corresponding marks.

1. Let $\mathbf{a} = (1, 4, 0)$ and $\mathbf{b} = (0, 1, 1)$. [5 marks]

(a) Calculate the dot product. [2 marks]

Add the products of matching components: 4.

Answer: 4

(b) Find their angle in degrees to one decimal place. [3 marks]

$\cos\theta = \frac{2\sqrt{34}}{17}$.

Use inverse cosine in degree mode.

Answer: 46.7°

2. Let $\mathbf{a} = (u, 4, 2)$, with u real. [4 marks]

(a) Find u if $|\mathbf{a}| = \sqrt{29}$. [2 marks]

Squaring the magnitude gives $u^2 + 20 = 29$.

Thus u squared is 9.

Answer: $u = 3$ or $u = -3$

(b) Find u if a is perpendicular to $(4, -1, 1)$. [2 marks]

The dot product is $4u - 4 + 2$.

Set the dot product to zero.

Answer: $u = \frac{1}{2}$

3. A triangular panel has vertices $O(0,0,0)$, $A(2,0,0)$, $B(0,k,1)$, where $k>0$. Its area is $\sqrt{10}$ square metres; coordinates are in metres. [7 marks]

(a) Find $\overrightarrow{OA}\times\overrightarrow{OB}$ in terms of k. [2 marks]

Apply the component formula to $(2,0,0)$ and $(0,k,1)$.

Answer: $(0,\ -2,2k)$.

(b) Use the area to find k. [3 marks]

The triangle area is half the cross-product magnitude: $\sqrt{1+k^2}$.

Thus $1+k^2=10$, and the positive condition gives $k=3$.

Answer: $k=3$.

(c) Find a unit normal with positive z component. [2 marks]

At $k=3$ the cross product is $(0,\ -2,6)$.

Divide it by its magnitude $2\sqrt{10}$.

Answer: $\frac{(0,\ -1,3)}{\sqrt{10}}$.

4. A cell is spanned by $\mathbf{a}=(1,1,0)$, $\mathbf{b}=(0,2,1)$, $\mathbf{c}=(4,0,3)$. [4 marks]

(a) Find the scalar triple product. [2 marks]

Compute $b\times c$, then dot with a to obtain 10.

Answer: 10

(b) Find its volume and say whether the vectors are coplanar. [2 marks]

Volume is the absolute triple product.

The nonzero result rules out coplanarity.

Answer: Volume 10 cubic units; not coplanar.

5. Planes $\Pi_1 : x + y = 4$ and $\Pi_2 : z = 2$ are given. [6 marks]

(a) Find a direction for their intersection line. [2 marks]

Cross normals $(1, 1, 0)$ and $(0, 0, 1)$.

Answer: $(1, -1, 0)$ or its nonzero multiples.

(b) Give a vector equation for the intersection. [2 marks]

Choose $y = t$.

Solve $x = 4 - t$ and keep $z = 2$.

Answer: $\mathbf{r} = (4, 0, 2) + t(-1, 1, 0)$

(c) Find the smaller angle between the planes. [2 marks]

Their normals have zero dot product.

Answer: 90°

6. Consider the family of lines $\ell_k : \mathbf{r} = (1, 0, 2) + t(1, 2, k)$ and plane $\Pi : x + 2y + 2z = 5$. [6 marks]

(a) Find k for which the line direction is parallel to the plane. [2 marks]

A normal is $(1, 2, 2)$.

Set its dot product with $(1, 2, k)$ to zero: $5 + 2k = 0$.

Answer: $k = -\frac{5}{2}$.

(b) For this value of k, decide whether the line is disjoint from the plane or contained in it. [2 marks]

The initial point $(1, 0, 2)$ satisfies $1 + 0 + 4 = 5$.

The direction is parallel to the plane, so every point remains in it.

Answer: The entire line is contained in the plane.

(c) Find k for which the line is perpendicular to the plane. [2 marks]

The direction must be a nonzero multiple of the normal $(1, 2, 2)$.

The first component fixes the multiple as 1, so $k = 2$.

Answer: $k = 2$.

7. Lines $\ell_1 = (s, 0, 0)$ and $\ell_2 = (t, 4, 4)$ model two straight rails. [4 marks]

(a) Explain why the cross-product skew-distance formula is unsuitable. [2 marks]

Both direction vectors are $(1, 0, 0)$.

Their cross product is zero.

Answer: The directions are parallel, giving a zero denominator.

(b) Find the rail separation. [2 marks]

A connecting vector perpendicular to the rails is $(0, 4, 4)$.

Its magnitude is $\sqrt{16+16} = 4\sqrt{2}$.

Answer: $4\sqrt{2}$

8. $P = (4, 2, 3)$ and plane $\Pi : x + y + z = 0$. [4 marks]

(a) Find the perpendicular foot Q on the plane. [2 marks]

Write $Q = P - \lambda(1, 1, 1)$.

The plane equation gives $\lambda = 3$.

Answer: $(1, -1, 0)$

(b) Find PQ and verify perpendicularity. [2 marks]

$Q - P$ is a scalar multiple of the plane normal.

Its length is $3\sqrt{3}$.

Answer: $3\sqrt{3}$

9. Consider $x+y+z=4$, $2x+2y+2z=8$, and $x-y=0$. [4 marks]

(a) Describe every solution. [2 marks]

Equation 2 repeats twice equation 1.

Set $x=y=t$ and solve for z.

Answer: $(t, t, 4-2t)$, real t.

(b) If the constant 8 in equation 2 is increased by 1, how does the solution set change? [2 marks]

Twice equation 1 would still give 8.

The changed equation contradicts it.

Answer: There is no solution.

10. Three unknown steady flow rates x, y and z, in litres per second, satisfy $x+y=30$, $y+z=25$, $x+z=35$. [7 marks]

(a) Write an augmented matrix for the system. [2 marks]

Use the order x, y, z for each row, keeping every equation constant in the augmented column.

Answer: $\begin{bmatrix} 1 & 1 & 0 & 30 \\ 0 & 1 & 1 & 25 \\ 1 & 0 & 1 & 35 \end{bmatrix}$.

(b) Solve for the three rates and check each original balance. [3 marks]

Add equations 1 and 2 and subtract equation 3: $2y=20$.

Then $x=30-10=20$ and $z=25-10=15$. The original sums are 30,25,35.

Answer: $x=20$, $y=10$, $z=15$ litres per second.

(c) Explain what a negative solution would mean for the assumed flow directions. [2 marks]

A signed rate is a direction convention; a negative value is not a negative amount of liquid.

Answer: The corresponding flow would run opposite to its assumed positive direction.