Vectors in three dimensions - practice test 3

10 questions | 58 marks | 70 minutes

Apply spatial methods and justify each conclusion.

Equivalent correct methods earn the corresponding marks.

1. A triangular frame has vertices $A(0,0,0)$, $B(6,0,0)$, $C(0,3,3)$. Its centroid G is defined to have position vector $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$. [6 marks]

(a) Find the midpoint M of BC. [2 marks]

Average the corresponding coordinates of B and C.

Answer: $M=(3,\frac{3}{2},\frac{3}{2})$.

(b) Find G and show that it lies on the median AM. [2 marks]

Average the three vertex position vectors.

The vector $(2,1,1)$ is two thirds of $(3,\frac{3}{2},\frac{3}{2})$, so G lies on AM.

Answer: $G=(2,1,1)$; $\overrightarrow{AG}=\frac{2}{3}\overrightarrow{AM}$.

(c) State $AG:GM$ and explain why this is internal division. [2 marks]

The parameter $\frac{2}{3}$ is between 0 and 1.

The two subsegment lengths are $\frac{2}{3}$ and $\frac{1}{3}$ of AM.

Answer: $AG:GM=2:1$.

2. A survey displacement is $\mathbf{a} = (3, 2, 2)$ metres. [4 marks]

(a) Find its exact length. [2 marks]

The sum of squared components is 17.

Take its positive square root.

Answer: $\sqrt{17}$ m

(b) Find a length-3 vector in the opposite direction. [2 marks]

The unit direction is $(\frac{3\sqrt{17}}{17}, \frac{2\sqrt{17}}{17}, \frac{2\sqrt{17}}{17})$.

Multiply this by -3.

Answer: $(-\frac{9\sqrt{17}}{17}, -\frac{6\sqrt{17}}{17}, -\frac{6\sqrt{17}}{17})$

3. Let $\mathbf{a} = (3, 2, 3)$ and $\mathbf{b} = (1, 1, 0)$. [5 marks]

(a) Find the vector projection of a along b. [2 marks]

Projection coefficient is $\frac{5}{2}$.

Multiply this scalar by b.

Answer: $(\frac{5}{2}, \frac{5}{2}, 0)$

(b) Find and verify the perpendicular component. [3 marks]

Subtract the projection from a. Dotting $(\frac{1}{2}, -\frac{1}{2}, 3)$ with b gives 0.

Answer: $(\frac{1}{2}, -\frac{1}{2}, 3)$

4. For arbitrary real vectors a and b, consider $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$. [7 marks]

(a) Expand and simplify this expression. [3 marks]

Distribute the dot product.

The mixed terms cancel because the dot product is commutative.

Answer: $|\mathbf{a}|^2-|\mathbf{b}|^2$

(b) Explain why the diagonals of a rhombus are perpendicular. [2 marks]

The diagonals are represented by $a+b$ and $a-b$.

Use the identity and $|a|=|b|$.

Answer: For equal-length adjacent sides a, b, their sum and difference have dot product 0.

(c) Check the identity for $a=(3,1,2)$, $b=(1,0,1)$. [2 marks]

Direct calculation gives 12.

Answer: Both sides equal 12.

5. $A=(1,0,3)$, $B=(2,1,3)$ and $C=(1,1,4)$ are three points. [6 marks]

(a) Find a nonzero normal to their plane. [2 marks]

Use $AB\times AC$.

The side directions are $(1,1,0)$ and $(0,1,1)$.

Answer: $(1,-1,1)$

(b) Write a Cartesian plane equation. [2 marks]

Use the normal $(1,-1,1)$.

The constant is $n\cdot A=4$.

Answer: $x-y+z=4$

(c) Check whether $P=(2,0,3)$ lies in the plane. [2 marks]

Increasing x by 1 increases the left side by 1.

The plane constant is unchanged.

Answer: No.

6. Two straight tracks are modelled by $\mathbf{r} = (1, 0, 2) + t(1, 2, -1)$ and $\mathbf{r} = (3, 1, 0) + s(-1, 1, 1)$, with real parameters. [7 marks]

(a) Find the intersection, using a separate parameter for each track. [3 marks]

The x and y equations give $t + s = 2$ and $2t - s = 1$.

Solve to get $t = s = 1$; the z coordinates both equal 1.

Answer: $t = s = 1$; intersection $(2, 2, 1)$.

(b) If t and s are each elapsed seconds after the same start time, do two objects following these equations collide? [2 marks]

A collision requires the same position at the same time.

The intersection parameters are equal, so both objects are at $(2, 2, 1)$ when the elapsed time is 1 second.

Answer: Yes, at 1 second.

(c) Explain why intersecting tracks alone would not guarantee a collision. [2 marks]

Independent line parameters locate points; they are not automatically equal elapsed times.

Answer: The objects may reach the intersection at different times.

7. Consider $\ell : \mathbf{r} = (0, 0, 3) + t(1, -1, a)$ and plane $x + y + z = 3$. [4 marks]

(a) For a nonzero, find the parameter of the intersection. [2 marks]

Substitution gives $3 + at = 3$, so $at = 0$.

Because a is nonzero, $t = 0$.

Answer: $t = 0$.

(b) Classify the case $a = 0$. [2 marks]

With $a = 0$, the left side of the plane equation stays 3.

Compare this constant with 3.

Answer: The entire line lies in the plane.

8. $P = (0, 3, 2)$ and $\ell : \mathbf{r} = (1, 0, 0) + t(1, 1, 0)$. [5 marks]

(a) Find the parameter of the closest point Q. [2 marks]

Perpendicularity gives $(1 + t, t - 3, -2) \cdot (1, 1, 0) = 0$.

This simplifies to $2t + 1 - 3 = 0$.

Answer: $t = 1$

(b) Find Q and the shortest distance. [3 marks]

Substitute the parameter into the line.

Calculate $|Q - P| = 2\sqrt{3}$.

Answer: $Q = (2, 1, 0)$, distance $2\sqrt{3}$.

9. Three planes have equations $x+y+z=1$, $x-y+z=3$ and $2x+2z=k$, where k is real. [7 marks]

(a) Eliminate y from the first two equations. What value of k allows a common solution? [2 marks]

Add the first two original equations.

Their sum must agree with the third equation.

Answer: $2x+2z=4$, so $k=4$.

(b) For that value of k, describe the entire common solution set. [3 marks]

Subtract equation 2 from equation 1 to obtain $2y=-2$.

Set $z=t$; then $x+z=2$ gives $x=2-t$.

Answer: $(x, y, z)=(2-t,\ -1, t)$, for real t.

(c) Describe the geometry when k is any other value. [2 marks]

The first two force $2x+2z=4$, which contradicts $2x+2z=k$ when k is not 4.

Answer: There is no point common to all three planes.

10. Consider lines $\ell_1=s(1,1,0)$ and $\ell_2=(0,3,1)+t(1,0,1)$. [7 marks]

(a) Show that the lines are skew. [2 marks]

An intersection would require $t=-1$ from the z coordinate and $s=-1$ from the x coordinate.

Then the y coordinate would require $-1=3$, which is impossible.

Answer: Their directions are not parallel and the lines do not intersect.

(b) Find the closest points P and Q, and their distance. [5 marks]

The joining vector is $(t-s, 3-s, 1+t)$. Perpendicularity gives $t+3-2s=0$ and $2t-s+1=0$.

Solve to obtain $s=\frac{5}{3}$ and $t=\frac{1}{3}$. Substitute in the two lines and calculate the joining-vector magnitude.

Answer: $P=(\frac{5}{3}, \frac{5}{3}, 0)$, $Q=(\frac{1}{3}, 3, \frac{4}{3})$; distance $\frac{4\sqrt{3}}{3}$.