Vectors in three dimensions - practice test 2

10 questions | 48 marks | 70 minutes

Connect algebra with geometric interpretation.

Equivalent correct methods earn the corresponding marks.

1. Let $\mathbf{a} = (u, 2, 2)$, with u real. [4 marks]

(a) Find u if $|\mathbf{a}| = \sqrt{17}$. [2 marks]

Squaring the magnitude gives $u^2 + 8 = 17$.

Thus u squared is 9.

Answer: $u = 3$ or $u = -3$

(b) Find u if a is perpendicular to $(2, -1, 1)$. [2 marks]

The dot product is $2u - 2 + 2$.

Set the dot product to zero.

Answer: $u = 0$

2. A constant force $\mathbf{F} = (6, 2, -1)$ N moves a component through displacement $(3, -1, 2)$ m. [4 marks]

(a) Find the work done. [2 marks]

Work is $F \cdot \text{displacement}$.

The dot product is $18 - 2 - 2 = 14$.

Answer: 14 J

(b) Resolve the force into a component along the x axis and a perpendicular component. [2 marks]

Project onto the unit x direction.

Subtract this projection from the force.

Answer: Along x: $(6, 0, 0)$ N; perpendicular: $(0, 2, -1)$ N.

3. A cell is spanned by $\mathbf{a} = (1, 1, 0)$, $\mathbf{b} = (0, 2, 1)$, $\mathbf{c} = (2, 0, 3)$. [4 marks]

(a) Find the scalar triple product. [2 marks]

Compute $b \times c$, then dot with a to obtain 8.

Answer: 8

(b) Find its volume and say whether the vectors are coplanar. [2 marks]

Volume is the absolute triple product.

The nonzero result rules out coplanarity.

Answer: Volume 8 cubic units; not coplanar.

4. Two sides of a triangular panel are $(2, 1, 0)$ m and $(0, 1, 2)$ m. [6 marks]

(a) Find their cross product. [2 marks]

Use the determinant/component formula.

The result is $(2, -4, 2)$.

Answer: $(2, -4, 2)$

(b) Find the panel area. [2 marks]

The parallelogram area is $2\sqrt{6}$.

Halve it for the triangle.

Answer: $\sqrt{6}\ \text{m}^2$

(c) Give a unit normal with positive x component. [2 marks]

The cross product has positive x component.

Divide it by its magnitude.

Answer: $(\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6})$

5. Planes $\Pi_1 : x + y = 2$ and $\Pi_2 : z = 2$ are given. [6 marks]

(a) Find a direction for their intersection line. [2 marks]

Cross normals $(1, 1, 0)$ and $(0, 0, 1)$.

Answer: $(1, -1, 0)$ or its nonzero multiples.

(b) Give a vector equation for the intersection. [2 marks]

Choose $y = t$.

Solve $x = 2 - t$ and keep $z = 2$.

Answer: $\mathbf{r} = (2, 0, 2) + t(-1, 1, 0)$

(c) Find the smaller angle between the planes. [2 marks]

Their normals have zero dot product.

Answer: 90°

6. Consider lines $\ell_1 = s(1, 1, 0)$ and $\ell_2 = (0, 2, 1) + t(1, 0, 1)$. [7 marks]

(a) Show that the lines are skew. [2 marks]

An intersection would require $t = -1$ from the z coordinate and $s = -1$ from the x coordinate.

Then the y coordinate would require $-1 = 2$, which is impossible.

Answer: Their directions are not parallel and the lines do not intersect.

(b) Find the closest points P and Q, and their distance. [5 marks]

The joining vector is $(t - s, 2 - s, 1 + t)$. Perpendicularity gives $t + 2 - 2s = 0$ and $2t - s + 1 = 0$.

Solve to obtain $s = 1$ and $t = 0$. Substitute in the two lines and calculate the joining-vector magnitude.

Answer: $P = (1, 1, 0)$, $Q = (0, 2, 1)$; distance $\sqrt{3}$.

7. Lines $\ell_1 = (s, 0, 0)$ and $\ell_2 = (t, 2, 4)$ model two straight rails. [4 marks]

(a) Explain why the cross-product skew-distance formula is unsuitable. [2 marks]

Both direction vectors are $(1, 0, 0)$.

Their cross product is zero.

Answer: The directions are parallel, giving a zero denominator.

(b) Find the rail separation. [2 marks]

A connecting vector perpendicular to the rails is $(0, 2, 4)$.

Its magnitude is $\sqrt{4+16} = 2\sqrt{5}$.

Answer: $2\sqrt{5}$

8. $P = (2, 2, 3)$ and plane $\Pi : x + y + z = 0$. [4 marks]

(a) Find the perpendicular foot Q on the plane. [2 marks]

Write $Q = P - \lambda(1, 1, 1)$.

The plane equation gives $\lambda = \frac{7}{3}$.

Answer: $(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$

(b) Find PQ and verify perpendicularity. [2 marks]

$Q - P$ is a scalar multiple of the plane normal.

Its length is $\frac{7\sqrt{3}}{3}$.

Answer: $\frac{7\sqrt{3}}{3}$

9. Consider $x+y+z=2$, $2x+2y+2z=4$, and $x-y=0$. [4 marks]

(a) Describe every solution. [2 marks]

Equation 2 repeats twice equation 1.

Set $x=y=t$ and solve for z.

Answer: $(t, t, 2-2t)$, real t.

(b) If the constant 4 in equation 2 is increased by 1, how does the solution set change? [2 marks]

Twice equation 1 would still give 4.

The changed equation contradicts it.

Answer: There is no solution.

10. An augmented system has rows $[1, 2, -1 \mid 4]$, $[3, 7, -2 \mid 13]$, $[0, 0, 1 \mid 2]$. [5 marks]

(a) Apply $R_2 \leftarrow R_2 - 3R_1$. [2 marks]

Subtract 3 times every entry of row 1, including the constant.

Answer: The new row 2 is $[0, 1, 1 \mid 1]$.

(b) Find the unique solution. [3 marks]

Row 3 gives $z=2$; row 2 gives $y=-1$.

Row 1 gives $x=4-2y+z=8$.

Answer: $(x, y, z)=(8, -1, 2)$.