Functions and graph sketching - Practice test 6

10 questions | 50 marks | 75 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x)=\sqrt{1-x}$ and $g(x)=\frac{x}{x-1}$. [5 marks]

(a) Find the rule and domain of $f\circ g$. [3 marks]

$1-\frac{x}{x-1}=-\frac{1}{x-1}=\frac{1}{1-x}$. The inner function excludes 1, and the root requires this positive expression.

Answer: $\sqrt{\frac{1}{1-x}}$ for $x<1$.

(b) Find the range and explain whether the composition is one-to-one. [2 marks]

As x increases from $-\infty$ to 1, $\frac{1}{1-x}$ rises from 0 to ∞, as does its square root.

Answer: Range $(0,\infty)$; it is strictly increasing and one-to-one.

2. Let $f(x)=\frac{x^2}{x^2+1}$ for $x\geq 0$. [5 marks]

(a) Find its range and prove that it is one-to-one on this domain. [2 marks]

$f=1-\frac{1}{x^2+1}$, which increases as nonnegative x increases; it starts at 0 and approaches 1.

Answer: Range $[0,1)$; it is strictly increasing.

(b) Find f^{-1}, with its domain and range. [3 marks]

From $y=\frac{x^2}{x^2+1}$ obtain $x^2=\frac{y}{1-y}$, and choose the nonnegative root.

Answer: $f^{-1}(x)=\sqrt{\frac{x}{1-x}}$; domain $[0,1)$, range $[0,\infty)$.

3. Let $f(x) = (x-2)^2 + 1$ restricted to $x \leq 2$. [7 marks]

(a) Find its inverse, stating all necessary restrictions. [3 marks]

The original input lies on the left branch, so $x - 2 \leq 0$.

Answer: $f^{-1}(x) = 2 - \sqrt{x-1}$ for $x \geq 1$; inverse range $(-\infty, 2]$.

(b) Find every point common to the graph and its inverse. [4 marks]

Check $f(1) = 2$ and $f(2) = 1$, giving the first two common points. For completeness, put $u = 2 - x \geq 0$ and $v = 2 - y \geq 0$. Mutual membership gives $1 - v = u^2$ and $1 - u = v^2$, hence $(u - v)(u + v - 1) = 0$. If $u = v$, then $u^2 + u - 1 = 0$, giving $u = v = \frac{\sqrt{5}-1}{2}$. If $u + v = 1$, then $u^2 = u$, giving $u = 0$ or $u = 1$. These yield exactly the three stated points.

Answer: $(1, 2)$, $(2, 1)$, and $(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2})$.

4. Consider $h(x) = |x^2 - 4|$. [5 marks]

(a) State the number of distinct solutions of $h(x) = c$ for $c < 0$, $c = 0$, $0 < c < 4$, $c = 4$ and $c > 4$. [3 marks]

Solve $x^2 = 4 + c$ and $x^2 = 4 - c$, counting only nonnegative right sides and avoiding duplication when $c = 0$.

Answer: Respectively $0, 2, 4, 3, 2$.

(b) Find the three intersections when $c = 4$ and identify the tangent intersection. [2 marks]

The central branch is $4 - x^2$, with horizontal tangent at zero. The outer branch gives $x^2 = 8$.

Answer: $(0, 4)$ and $(\pm 2\sqrt{2}, 4)$; the line is tangent at $(0, 4)$.

5. A nonzero ordinate y is transformed by $T(y)=\frac{1}{y}$. [4 marks]

(a) Find its fixed values and show that applying T twice restores every allowed ordinate. [2 marks]

$y=\frac{1}{y}$ gives $y^2=1$; $\frac{1}{\frac{1}{y}}=y$.

Answer: Fixed values ±1; $T(T(y))=y$ for $y\neq0$.

(b) For $f(x)=x^2-1$, determine where $f(x)$ and $\frac{1}{f(x)}$ are both positive and where they are both negative. [2 marks]

Reciprocation preserves sign. The roots ±1 are excluded from the reciprocal.

Answer: Both positive for $x<-1$ or $x>1$; both negative for $-1<x<1$.

6. Let $F(x)=\frac{x^2-2x+1}{x^2-1}$. [5 marks]

(a) Simplify and identify all holes, poles and intercepts. [3 marks]

Cancel one factor $x-1$. The only numerator zero is still excluded from the original domain.

Answer: $F=\frac{x-1}{x+1}$ for $x\neq\pm1$; hole $(1,0)$; pole $x=-1$; vertical intercept $(0,-1)$ and no horizontal intercept.

(b) Find the range and explain why two values are missing. [2 marks]

The simplified rule is $1-\frac{2}{x+1}$ and never equals 1. Its only zero would be the hole input 1.

Answer: $\mathbb{R}\setminus\{0,1\}$.

7. Let $f(x)=\frac{x^2}{x-1}$ and consider the intersections with $y=c$. [5 marks]

(a) Derive the discriminant condition for a real intersection. [3 marks]

$x^2-cx+c=0$ has discriminant c^2-4c. The excluded input 1 never satisfies this equation.

Answer: $c(c-4)\geq0$, so $c\leq0$ or $c\geq4$.

(b) State how many distinct intersections occur in each permitted range, including the boundary cases. [2 marks]

Positive discriminant gives two roots, zero discriminant a repeated tangent root, negative discriminant no real roots.

Answer: Two when $c<0$ or $c>4$; one at $c=0$ or $c=4$; none when $0<c<4$.

8. For $a > 0$, define $f_a(x) = \frac{x^2 - a^2}{x^2 + a^2}$. [5 marks]

(a) Find the zeros, stationary point, asymptote and range in terms of a. [3 marks]

$f_a = 1 - \frac{2a^2}{x^2 + a^2}$; $f_a' = \frac{4a^2 x}{(x^2 + a^2)^2}$, with a negative-to-positive sign change at zero.

Answer: Zeros $\pm a$; global minimum $(0, -1)$; horizontal asymptote $y = 1$; range $[-1, 1)$.

(b) Describe precisely how its graph is obtained from the case $a = 1$. [2 marks]

Writing $f_a(x) = f_1(\frac{x}{a})$ shows that every horizontal coordinate is multiplied by a; vertical coordinates remain the same.

Answer: A horizontal stretch by factor a.

9. A cost per unit model is $C(x) = 2 + \frac{18}{x - 1}$ for $x > 1$. [5 marks]

(a) Find the batch size for cost 5 dollars. [3 marks]

$\frac{18}{x - 1} = 3$, so $x - 1 = 6$.

Answer: $x = 7$.

(b) Explain the horizontal asymptote in context. [2 marks]

The extra term tends to zero but stays positive.

Answer: Cost tends toward 2 dollars per unit as batch size increases.

10. A student labels $(0, 0)$ as a maximum on the curve $y = \frac{x^3}{x^2 - 1}$. [4 marks]

(a) Use the derivative to correct the label. [3 marks]

$f'(x) = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2}$, so $f'(0) = 0$. [1 mark]

The first derivative is negative immediately on both sides of zero, so the point is neither a local maximum nor a local minimum. [1 mark]

$f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$ is positive just left of zero and negative just right. Concavity changes, so $(0, 0)$ is a stationary inflection. [1 mark]

Answer: It is a stationary inflection, not a maximum.

(b) Identify two excluded inputs the sketch must respect. [1 mark]

The denominator vanishes there.

Answer: $x = -1$ and $x = 1$.