Functions and graph sketching - Practice test 5

10 questions | 51 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x) = \ln x$ and $g(x) = 1 - x^2$. [5 marks]

(a) Find the domains of both compositions. [2 marks]

The first logarithm needs $1 - x^2 > 0$; in the second composition the inner logarithm alone restricts the input.

Answer: $f \circ g$: $-1 < x < 1$; $g \circ f$: $x > 0$.

(b) Find and compare their ranges. [3 marks]

On $(-1, 1)$, $1 - x^2$ ranges over $(0, 1]$. Meanwhile $\ln x$ ranges over all real numbers, so $1 - (\ln x)^2 \leq 1$.

Answer: $f \circ g$ has range $(-\infty, 0]$; $g \circ f$ has range $(-\infty, 1]$.

2. Let $f(x) = x^3 - 3x$ restricted to $[-1, 1]$. [5 marks]

(a) Justify the existence of an inverse and state its domain and range, without finding an explicit rule. [3 marks]

$f' = 3x^2 - 3 < 0$ in the interior; $f(-1) = 2$ and $f(1) = -2$.

Answer: The function is strictly decreasing; inverse domain $[-2, 2]$, inverse range $[-1, 1]$.

(b) Find the inverse images of $-2, 0, 2$. [2 marks]

Use the values of the original function and the restricted domain.

Answer: $f^{-1}(-2) = 1$, $f^{-1}(0) = 0$, $f^{-1}(2) = -1$.

3. Let $f(x) = 4 - \sqrt{2x+1}$ with its largest real domain. [5 marks]

(a) Find the inverse and its domain and range. [3 marks]

The original root requires $x \geq -\frac{1}{2}$ and gives outputs at most 4. Solve $\sqrt{2x+1} = 4 - y$, keeping $y \leq 4$ before squaring.

Answer: $f^{-1}(x) = \frac{(4-x)^2 - 1}{2}$; domain $(-\infty, 4]$, range $[-\frac{1}{2}, \infty)$.

(b) Explain why the inverse formula must not be used at input 5. [2 marks]

Squaring the rearranged equation alone loses the condition $4 - y \geq 0$.

Answer: 5 lies outside the inverse domain because the original function never outputs 5.

4. Let $f(x) = x^2 - 2$. Compare $A(x) = |f(x)|$ and $B(x) = f(|x|)$. [4 marks]

(a) Find the ranges and vertical intercepts of A and B. [2 marks]

Squaring removes the input sign in B, whereas A reflects negative outputs upward.

Answer: A: range $[0, \infty)$, vertical intercept 2. $B = x^2 - 2$: range $[-2, \infty)$, vertical intercept -2.

(b) Find exactly where the two graphs agree. [2 marks]

Equality $|x^2 - 2| = x^2 - 2$ holds precisely where $x^2 - 2 \geq 0$.

Answer: $x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$.

5. Let $f(x) = \frac{x}{x^2+1}$ and $r(x) = \frac{1}{f(x)}$. [5 marks]

(a) Find r, its original domain and its range. [3 marks]

The original zero $f(0) = 0$ is excluded. The positive branch has minimum 2 at 1; odd symmetry gives the negative range.

Answer: $r = x + \frac{1}{x}$, $x \neq 0$; range $(-\infty, -2] \cup [2, \infty)$.

(b) Explain why no real point of f is fixed by ordinate reciprocation. [2 marks]

Solve $y = \frac{1}{y}$ to get $y = \pm 1$, or use the known extrema of f.

Answer: The range of f is $[-\frac{1}{2}, \frac{1}{2}]$, so it never reaches the fixed ordinates 1 or -1.

6. Let $f(x)=\frac{x^2+ax+b}{x-1}$, where a, b are real. [5 marks]

(a) Find the condition on a, b for the graph to have a hole rather than a vertical asymptote at $x=1$. [2 marks]

The numerator must also vanish at 1; then it contains a factor $x-1$.

Answer: $1+a+b=0$.

(b) If the hole is $(1, 4)$, determine a, b and the simplified rule. [3 marks]

Under the condition in part (a), the quotient is $x+a+1$. Its missing value at 1 is $a+2=4$, giving $a=2$ and $b=-3$.

Answer: $a=2, b=-3$; $f=x+3$ for $x\neq 1$.

7. Let $f(x)=\frac{x^3}{x^2-1}$. [6 marks]

(a) Find its asymptotes and all stationary inputs. [3 marks]

Division gives $x+\frac{x}{x^2-1}$; $f'=\frac{x^2(x^2-3)}{(x^2-1)^2}$.

Answer: Asymptotes $x=\pm 1$, $y=x$; stationary inputs $0, \pm\sqrt{3}$.

(b) Classify the stationary points and give their coordinates. [3 marks]

At $x=-\sqrt{3}$, f' changes from positive to negative; substituting gives the local maximum $(-\sqrt{3}, -\frac{3\sqrt{3}}{2})$. [1 mark]

At $x=\sqrt{3}$, f' changes from negative to positive; substituting gives the local minimum $(\sqrt{3}, \frac{3\sqrt{3}}{2})$. [1 mark]

At zero, $f'(0)=0$. Also $f''(x)=\frac{2x(x^2+3)}{(x^2-1)^3}$ changes from positive to negative there, proving the stationary inflection $(0, 0)$. [1 mark]

Answer: Local maximum $(-\sqrt{3}, -\frac{3\sqrt{3}}{2})$; stationary inflection $(0, 0)$; local minimum $(\sqrt{3}, \frac{3\sqrt{3}}{2})$.

8. Consider the family $f_a(x) = x + \frac{a}{x}$ on $x \neq 0$. [6 marks]

(a) Find the stationary points when $a > 0$, and show that none exist when $a < 0$. [3 marks]

$f_a' = 1 - \frac{a}{x^2}$. The equation $x^2 = a$ has real nonzero solutions only for positive a.

Answer: For $a > 0$: local maximum $(-\sqrt{a}, -2\sqrt{a})$ and local minimum $(\sqrt{a}, 2\sqrt{a})$. For $a < 0$: no stationary point.

(b) Compare the ranges for $a = 0$ and $a < 0$. [3 marks]

For zero parameter the rule is x with its original missing input. For negative parameter each branch is strictly increasing and runs from $-\infty$ to ∞ between its endpoint limits.

Answer: $a = 0$: $\mathbb{R} \setminus \{0\}$. $a < 0$: all real numbers.

9. A concentration model is $c(t) = \frac{t}{t^2 + 4}$ for $t \geq 0$. [6 marks]

(a) Find its maximum. [4 marks]

$c'(t) = \frac{4 - t^2}{(t^2 + 4)^2}$.

The derivative changes from positive to negative at $t = 2$.

Answer: Maximum $\frac{1}{4}$ at $t = 2$.

(b) State the long-term prediction and a limitation. [2 marks]

The denominator grows quadratically while the numerator grows linearly.

Answer: Concentration approaches zero; the idealised rule need not remain accurate indefinitely.

10. A student chooses both branches $3 \pm \sqrt{x - 2}$ as the inverse of $(x - 3)^2 + 2$ restricted to $x \geq 3$. [4 marks]

(a) Correct the inverse. [2 marks]

The original allowed input minus 3 is nonnegative.

Answer: $3 + \sqrt{x - 2}$ only.

(b) Explain why both branches cannot be used. [2 marks]

An inverse function must be single-valued.

Answer: The result would assign two outputs to some inverse inputs.