Functions and graph sketching - Practice test 4

10 questions | 48 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x)=\sqrt{4-x}$ and $g(x)=\frac{1}{x}$. [5 marks]

(a) Find the domain of $f\circ g$. [3 marks]

Require $x\neq 0$ and $4-\frac{1}{x}\geq 0$. Every negative input works; for positive inputs multiply by x to obtain $x\geq\frac{1}{4}$.

Answer: $(-\infty,0)\cup[\frac{1}{4},\infty)$.

(b) Find the domain of $g\circ f$. Why does an endpoint disappear? [2 marks]

The root accepts $x\leq 4$, but at 4 it outputs zero, which the reciprocal cannot accept.

Answer: $x<4$.

2. Consider the family $f_a(x)=ax^2+2x$, with real a. [5 marks]

(a) For which a is f_a one-to-one on the entire real line? [2 marks]

When $a=0$, the function is the line $2x$. If $a\neq 0$, its quadratic graph is symmetric about $x=-\frac{1}{a}$ and fails the horizontal-line test.

Answer: Only $a=0$.

(b) For $a=1$, restrict to the increasing branch and give its inverse. [3 marks]

$x^2+2x=(x+1)^2-1$; choose the nonnegative square-root branch.

Answer: Domain $[-1,\infty)$; inverse $-1+\sqrt{x+1}$ with inverse domain $[-1,\infty)$.

3. Let $f(x)=\frac{x}{x-1}$ on $\mathbb{R}\setminus\{1\}$. [4 marks]

(a) Show that f is its own inverse. [2 marks]

$f(f(x))=\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}=x$. The inner value cannot equal 1.

Answer: $f(f(x))=x$ for every $x\neq 1$.

(b) Find its real fixed points and explain their relation to $y=x$. [2 marks]

Solve $\frac{x}{x-1}=x$: $x(x-2)=0$. Both are allowed, and each lies on the reflection line.

Answer: $(0,0)$ and $(2,2)$.

4. Let $f(x)=|x^2-2x|$. [5 marks]

(a) Give its piecewise rule and all local extrema. [3 marks]

The quadratic inside is nonpositive on $[0,2]$ and nonnegative outside.

Answer: x^2-2x for $x\leq 0$ or $x\geq 2$, and $2x-x^2$ for $0\leq x\leq 2$; minima $(0,0),(2,0)$ and local maximum $(1,1)$.

(b) For which real c does $f(x)=c$ have four distinct solutions? [2 marks]

Each positive level below the middle maximum meets the two inner arcs and the two outer branches. At $c=1$ the two inner intersections merge.

Answer: $0<c<1$.

5. Let $r(x)=\frac{1}{x^2+2x+2}$. [5 marks]

(a) Find its domain, symmetry, maximum and range. [3 marks]

Complete the square in the denominator: $(x+1)^2+1$.

Answer: All real inputs; symmetric about $x=-1$; maximum $(-1,1)$; range $(0,1]$.

(b) Find all intersections of the graph with $y=\frac{1}{2}$. [2 marks]

The denominator must be 2, so $(x+1)^2=1$.

Answer: $(-2,\frac{1}{2})$ and $(0,\frac{1}{2})$.

6. For $f(x)=\frac{x^3-x}{x^2-1}$: [4 marks]

(a) Simplify, state the domain, and identify every hole. [2 marks]

Factor the numerator as $x(x-1)(x+1)$ and keep the original denominator exclusions.

Answer: $f(x)=x$ for $x\neq\pm1$; holes $(-1,-1)$ and $(1,1)$.

(b) Find the range and decide whether the function is one-to-one. [2 marks]

On the allowed domain the output equals the input, so neither omitted output can be obtained by another input.

Answer: Range $\mathbb{R}\setminus\{-1,1\}$; it is one-to-one.

7. Let $f(x)=\frac{x^2-1}{x^2+1}$. [5 marks]

(a) Find its intercepts, stationary point and range. [3 marks]

$f=1-\frac{2}{x^2+1}$ and $f'=\frac{4x}{(x^2+1)^2}$.

Answer: Intercepts $(\pm1,0),(0,-1)$; global minimum $(0,-1)$; range $[-1,1)$.

(b) How many intersections with a horizontal line $y=c$ occur for $-1<c<1$? Justify. [2 marks]

The graph is even and strictly decreases to its minimum then strictly increases; each branch approaches 1.

Answer: Two, one on each side of zero.

8. A rational graph has rule $y = x + \frac{4}{x-1}$. [6 marks]

(a) Find its asymptotes and stationary points. [3 marks]

$y' = 1 - \frac{4}{(x-1)^2} = 0$ gives $x = -1, 3$. Its signs classify the points.

Answer: Asymptotes $x = 1$, $y = x$; local maximum $(-1, -3)$ and local minimum $(3, 5)$.

(b) Justify its range using a quadratic discriminant. [3 marks]

$y(x-1) = x(x-1) + 4$ gives $x^2 - (y+1)x + y + 4 = 0$. Its discriminant is $(y-5)(y+3)$, which must be nonnegative.

Answer: $(-\infty, -3] \cup [5, \infty)$.

9. The height model $h(x) = |x-4| + 2$ is used on $0 \le x \le 8$. [5 marks]

(a) Find the minimum and its location. [2 marks]

The absolute-value part is smallest at zero.

Answer: Minimum 2 at $x = 4$.

(b) For which inputs is height 5? [3 marks]

Solve $|x-4| = 3$ and check both inputs lie in the domain.

Answer: $x = 1$ or $x = 7$.

10. A graph of $\frac{x}{x^2+1}$ is said never to cross $y = 0$ because $y = 0$ is its asymptote. [4 marks]

(a) Refute the claim. [2 marks]

The denominator at zero is 1, so this input is valid.

Answer: $f(0) = 0$.

(b) State what a horizontal asymptote actually describes. [2 marks]

It does not forbid intersections at finite inputs.

Answer: The limit at large positive or negative input magnitude.