Functions and graph sketching - Practice test 2

10 questions | 48 marks | 70 minutes

Ten multipart questions combining exact calculations, domains, graph evidence, interpretation and reasoning. Show working and label key graph features.

Equivalent correct methods earn the corresponding marks.

1. Let $f(x)=\frac{1}{x-1}$ and $g(x)=\sqrt{x}$. [4 marks]

(a) Find $(f\circ g)(x)$ and its domain. [2 marks]

The root requires $x\geq 0$; its value must also differ from 1.

Answer: $\frac{1}{\sqrt{x}-1}$, $x\geq 0$ and $x\neq 1$.

(b) Find $(g\circ f)(x)$ and its domain. [2 marks]

The reciprocal is defined when $x\neq 1$. Its value is nonnegative only when $x>1$; it cannot equal zero.

Answer: $\sqrt{\frac{1}{x-1}}$, $x>1$.

2. Consider $f(x)=(x-1)^2$ on the restricted domain $[-1,1]$. [5 marks]

(a) Prove that this restriction is one-to-one. [2 marks]

For inputs at most 1, increasing the input decreases its nonnegative distance from 1, and hence decreases its square. Equivalently, $f'(x)=2(x-1)<0$ in the interval interior.

Answer: It is strictly decreasing on $[-1,1]$.

(b) Find the inverse, including its domain and range. [3 marks]

The original input satisfies $x-1\leq 0$, selecting the negative root. The original endpoint values are 4 and 0.

Answer: $f^{-1}(x)=1-\sqrt{x}$; domain $[0,4]$, range $[-1,1]$.

3. Let $f(x)=\frac{2x+3}{x-1}$, $x\neq 1$. [4 marks]

(a) Find its inverse and excluded inverse input. [2 marks]

$y(x-1)=2x+3$ gives $x(y-2)=y+3$.

Answer: $f^{-1}(x)=\frac{x+3}{x-2}$, $x\neq 2$.

(b) Find $f^{-1}(-3)$ and verify the result in the original rule. [2 marks]

Substitution gives $\frac{-3+3}{-3-2}=0$, then $\frac{3}{-1}=-3$.

Answer: $f^{-1}(-3)=0$ and $f(0)=-3$.

4. Let $v(x)=|x^2-9|$. [5 marks]

(a) Find the zeros, local extrema and range. [3 marks]

Between -3 and 3 the graph is $9-x^2$; outside, it is x^2-9.

Answer: Zeros ± 3; corner minima $(\pm 3, 0)$; local maximum $(0, 9)$; range $[0, \infty)$.

(b) Find all intersections with $y=5$. [2 marks]

$x^2-9=5$ gives $x^2=14$; $x^2-9=-5$ gives $x^2=4$.

Answer: $x=\pm 2,\ \pm\sqrt{14}$.

5. Let $r(x)=-\frac{3}{x+2}+1$. [5 marks]

(a) Find its asymptotes, centre of symmetry and intercepts. [3 marks]

Set the rule equal to zero for the root; substitute $x=0$ for the vertical intercept.

Answer: Asymptotes $x=-2$, $y=1$; centre $(-2, 1)$; intercepts $(1, 0)$ and $(0, -\frac{1}{2})$.

(b) Determine where the graph is increasing and explain its range. [2 marks]

$r'=\frac{3}{(x+2)^2}>0$ wherever defined. The nonzero reciprocal term can have every nonzero real value.

Answer: Increasing on each of $(-\infty, -2)$ and $(-2, \infty)$; range $\mathbb{R}\setminus\{1\}$.

6. Let $f(x)=\frac{x+2}{x^2+x-2}$. [5 marks]

(a) Factor, simplify and locate every discontinuity. [3 marks]

The denominator is $(x+2)(x-1)$. Keep both original exclusions.

Answer: $f(x)=\frac{1}{x-1}$ for $x\neq -2, 1$; hole $(-2, -\frac{1}{3})$ and vertical asymptote $x=1$.

(b) Find the range, taking account of the hole. [2 marks]

The reciprocal rule never equals zero. It attains $-\frac{1}{3}$ only at the excluded input -2.

Answer: $\mathbb{R}\setminus\{0, -\frac{1}{3}\}$.

7. Let $f(x)=\frac{x^2}{x+2}$. [5 marks]

(a) Use division to find its asymptotes. [2 marks]

Divide the numerator by $x+2$; the remainder is 4.

Answer: $f(x)=x-2+\frac{4}{x+2}$; asymptotes $x=-2$, $y=x-2$.

(b) Find and classify its stationary points. [3 marks]

$f'=\frac{x(x+4)}{(x+2)^2}$. Its sign is positive before -4, negative between -4 and 0 apart from the pole, and positive after 0.

Answer: Local maximum $(-4, -8)$; local minimum $(0, 0)$.

8. Let $f(x)=\frac{x^2+1}{x^2+4}$. [5 marks]

(a) Find its symmetry, stationary point and range. [3 marks]

$f=1-\frac{3}{x^2+4}$ and $f'=\frac{6x}{(x^2+4)^2}$. The minimum occurs at zero and 1 is approached from below.

Answer: Even; global minimum $(0,\frac{1}{4})$; range $[\frac{1}{4},1)$.

(b) Give all intercepts and asymptotes needed for a sketch. [2 marks]

Neither numerator nor denominator has a real zero.

Answer: Vertical intercept $\frac{1}{4}$; no horizontal-axis intercept; horizontal asymptote $y=1$; no vertical asymptote.

9. A machine is calibrated by $y=(x-4)^2+1$ on $x\geq 4$. [5 marks]

(a) Find the inverse calibration and its domain. [3 marks]

Choose the nonnegative square-root branch.

Answer: $x=4+\sqrt{y-1}$, $y\geq 1$.

(b) Find the input producing output 10. [2 marks]

$4+\sqrt{9}=7$.

Answer: $x=7$.

10. A sketch of $\frac{1}{x^2+1}$ shows two vertical asymptotes. [5 marks]

(a) Explain why that cannot be correct. [2 marks]

x^2+1 is always at least one.

Answer: The real denominator never vanishes.

(b) State its actual domain, range and end behaviour. [3 marks]

The maximum occurs at zero.

Answer: Domain real; range $(0,1]$; approaches zero at both ends.