Complex Numbers - Paper 6

Name: _____ Date: _____

10 questions | 68 marks | 75 minutes

A challenging synthesis paper with exact roots, geometric restrictions, proof and polynomial reconstruction.

A scientific calculator may be used.

1. Let $z = -1 + \sqrt{3}i$. [6 marks]

(a) Find its principal-argument polar form and the principal argument of its inverse. [3 marks]

(b) Calculate z^5 exactly. [3 marks]

2. Two nonzero complex numbers satisfy $|u| = 3$, $|v| = 2$, $\text{Arg}\, u = \frac{5\pi}{6}$, and $\text{Arg}\, v = -\frac{2\pi}{3}$. [6 marks]

(a) Find the principal-argument polar form of $\frac{u}{v}$. [3 marks]

(b) Find $|u - v|$ exactly. [3 marks]

3. Let $z = x + iy$. Consider $|z - 1| = 2|z + 1|$. [6 marks]

(a) Find its Cartesian equation in completed-square form. [4 marks]

(b) Give the centre and radius and explain why the locus is not a line. [2 marks]

4. Solve $(z-1)^3 = 8i$. [6 marks]

(a) List all solutions in exact Cartesian form. [4 marks]

(b) State the centre of the triangle formed by the roots and justify it. [2 marks]

5. Let $\zeta = \text{cis}(\frac{2\pi}{5})$. [8 marks]

(a) Show that $\zeta + \zeta^{-1} = 2\cos(\frac{2\pi}{5})$ and is real. [2 marks]

(b) Put $t = \zeta + \zeta^{-1}$. Prove $t^2 + t - 1 = 0$. [4 marks]

(c) Hence find $\cos(\frac{2\pi}{5})$ exactly. [2 marks]

6. The locus $|z|=1$ is restricted by $\operatorname{Arg}\left(\frac{z-1}{z+1}\right)=\frac{\pi}{2}$. [7 marks]

(a) For $z=x+iy$ on the circle with $z \neq -1$, show that $\frac{z-1}{z+1}=\frac{2iy}{(x+1)^2+y^2}$. [4 marks]

(b) Describe exactly which part of the unit circle satisfies the argument condition. [3 marks]

7. Consider $P(z)=z^4-2z^3+6z^2-2z+5$. [6 marks]

(a) Show that $1+2i$ is a root by identifying a real quadratic factor, and find the quotient. [3 marks]

(b) Factorise P completely over the complex numbers. [3 marks]

8. A real monic cubic has roots $a+bi$, $a-bi$ and a real number r, with $b \neq 0$. Its coefficient of z^2 is 0 and its constant term is -20. Also $a=1$. [8 marks]

(a) Find r and explain whether a real value of b can satisfy all conditions. [4 marks]

(b) If the constant term were $+20$ instead, find b and the cubic. [4 marks]

9. A map is $T(z) = az + b$. It sends 0 to $1 - i$ and 1 to $3 + i$. [8 marks]

(a) Find a and b, and describe the scale and turn in its multiplication step. [4 marks]

(b) Find the unique fixed point of T. [4 marks]

10. Let n be an integer greater than 1 and $\omega = \text{cis}(\frac{2\pi}{n})$. [7 marks]

(a) Prove that the product of all n roots of unity is $(-1)^{n-1}$. [4 marks]

(b) Explain why conjugate pairing gives the same result, distinguishing even and odd n. [3 marks]