Complex Numbers - Paper 6

10 questions | 68 marks | 75 minutes

A challenging synthesis paper with exact roots, geometric restrictions, proof and polynomial reconstruction.

Equivalent correct methods earn the corresponding marks.

1. Let $z = -1 + \sqrt{3}\,i$. [6 marks]

(a) Find its principal-argument polar form and the principal argument of its inverse. [3 marks]

The point is in quadrant II; taking a reciprocal negates the argument.

Answer: $z = 2\text{cis}(\frac{2\pi}{3})$; $\text{Arg}(\frac{1}{z}) = -\frac{2\pi}{3}$.

(b) Calculate z^5 exactly. [3 marks]

De Moivre gives $32\text{cis}(\frac{10\pi}{3}) = 32\text{cis}(-\frac{2\pi}{3})$.

Answer: $-16 - 16\sqrt{3}\,i$.

2. Two nonzero complex numbers satisfy $|u| = 3$, $|v| = 2$, $\text{Arg}\,u = \frac{5\pi}{6}$, and $\text{Arg}\,v = -\frac{2\pi}{3}$. [6 marks]

(a) Find the principal-argument polar form of $\frac{u}{v}$. [3 marks]

Subtract angles: $\frac{5\pi}{6} + \frac{2\pi}{3} = \frac{3\pi}{2}$, which reduces to $-\frac{\pi}{2}$.

Answer: $\frac{3}{2}\text{cis}(-\frac{\pi}{2})$.

(b) Find $|u - v|$ exactly. [3 marks]

The angle difference has cosine 0. Thus $|u - v|^2 = |u|^2 + |v|^2 - 2|u||v|\cos(\frac{3\pi}{2}) = 9 + 4 = 13$.

Answer: $\sqrt{13}$.

3. Let $z = x + iy$. Consider $|z - 1| = 2|z + 1|$. [6 marks]

(a) Find its Cartesian equation in completed-square form. [4 marks]

Square: $(x-1)^2 + y^2 = 4[(x+1)^2 + y^2]$. Rearrange to $3x^2 + 10x + 3y^2 + 3 = 0$, then complete the square.

Answer: $(x + \frac{5}{3})^2 + y^2 = \frac{16}{9}$.

(b) Give the centre and radius and explain why the locus is not a line. [2 marks]

The unequal distance ratio leaves quadratic terms after expansion; it is a circle rather than an equal-distance bisector.

Answer: Centre $(-\frac{5}{3}, 0)$; radius $\frac{4}{3}$.

4. Solve $(z-1)^3 = 8i$. [6 marks]

(a) List all solutions in exact Cartesian form. [4 marks]

Let $w = z - 1$. The roots of $w^3 = 8i$ have modulus 2 and arguments $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$. Convert and add 1.

Answer: $z = 1 + \sqrt{3} + i,\ 1 - \sqrt{3} + i,\ 1 - 2i.$

(b) State the centre of the triangle formed by the roots and justify it. [2 marks]

The w-roots form a regular triangle centred at zero, and the map $z = w + 1$ translates every point by 1.

Answer: The centre is $1 + 0i$.

5. Let $\zeta = \operatorname{cis}(\frac{2\pi}{5})$. [8 marks]

(a) Show that $\zeta + \zeta^{-1} = 2\cos(\frac{2\pi}{5})$ and is real. [2 marks]

The two terms are conjugates on the unit circle, so their imaginary parts cancel.

Answer: $\zeta + \zeta^{-1} = 2\cos(\frac{2\pi}{5})$.

(b) Put $t = \zeta + \zeta^{-1}$. Prove $t^2 + t - 1 = 0$. [4 marks]

Divide $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$ by ζ^2: $(\zeta^2 + \zeta^{-2}) + (\zeta + \zeta^{-1}) + 1 = 0$. Since $\zeta^2 + \zeta^{-2} = t^2 - 2$, obtain the quadratic.

Answer: $t^2 + t - 1 = 0$.

(c) Hence find $\cos(\frac{2\pi}{5})$ exactly. [2 marks]

$t = \frac{-1 \pm \sqrt{5}}{2}$; $\cos(\frac{2\pi}{5}) > 0$, so take the positive t and divide by 2.

Answer: $\frac{\sqrt{5} - 1}{4}$.

6. The locus $|z|=1$ is restricted by $\operatorname{Arg}\left(\frac{z-1}{z+1}\right)=\frac{\pi}{2}$. [7 marks]

(a) For $z=x+iy$ on the circle with $z\neq -1$, show that $\frac{z-1}{z+1}=\frac{2iy}{(x+1)^2+y^2}$. [4 marks]

Multiply by $\overline{z+1}=x+1-iy$. The numerator is $x^2+y^2-1+2iy=2iy$ on the unit circle.

Answer: The quotient has the stated pure-imaginary form.

(b) Describe exactly which part of the unit circle satisfies the argument condition. [3 marks]

The denominator above is positive where defined, so the quotient has argument $\frac{\pi}{2}$ exactly when $y>0$. At $z=-1$ it is undefined; at $z=1$ it is zero with undefined argument.

Answer: The open upper semicircle, $|z|=1$ with $y>0$; both real endpoints are excluded.

7. Consider $P(z)=z^4-2z^3+6z^2-2z+5$. [6 marks]

(a) Show that $1+2i$ is a root by identifying a real quadratic factor, and find the quotient. [3 marks]

The pair $1\pm 2i$ yields z^2-2z+5; multiplication by z^2+1 gives the stated P.

Answer: $P(z)=(z^2-2z+5)(z^2+1)$.

(b) Factorise P completely over the complex numbers. [3 marks]

The quotient has roots $\pm i$ and the first factor has roots $1\pm 2i$.

Answer: $(z-(1+2i))(z-(1-2i))(z-i)(z+i)$.

8. A real monic cubic has roots $a+bi$, $a-bi$ and a real number r, with $b \neq 0$. Its coefficient of z^2 is 0 and its constant term is -20. Also $a=1$. [8 marks]

(a) Find r and explain whether a real value of b can satisfy all conditions. [4 marks]

The root sum is $2a+r=0$, so $r=-2$. The constant is $-r(a^2+b^2)=2(1+b^2)>0$, which cannot be -20.

Answer: $r=-2$; no real b can satisfy the constant-term condition.

(b) If the constant term were $+20$ instead, find b and the cubic. [4 marks]

$2(1+b^2)=20$ gives $b^2=9$. Multiply $(z+2)(z^2-2z+10)$ to obtain $z^3+6z+20$.

Answer: $b=\pm 3$; $P(z)=z^3+6z+20$.

9. A map is $T(z)=az+b$. It sends 0 to $1-i$ and 1 to $3+i$. [8 marks]

(a) Find a and b, and describe the scale and turn in its multiplication step. [4 marks]

$T(0)=b$. Then $a+b=3+i$, giving $a=2+2i$. Express a in polar form.

Answer: $b=1-i$, $a=2+2i$; scale $2\sqrt{2}$, anticlockwise rotation $\frac{\pi}{4}$.

(b) Find the unique fixed point of T. [4 marks]

From $z=(2+2i)z+1-i$, obtain $(-1-2i)z=1-i$. Divide with the conjugate: $z=\frac{(1-i)(-1+2i)}{5}=\frac{1+3i}{5}$.

Answer: $z=\frac{1+3i}{5}$.

10. Let n be an integer greater than 1 and $\omega=\operatorname{cis}(\frac{2\pi}{n})$. [7 marks]

(a) Prove that the product of all n roots of unity is $(-1)^{n-1}$. [4 marks]

The product is $\omega^{0+1+\cdots+(n-1)}=\operatorname{cis}(\pi(n-1))=(-1)^{n-1}$.

Answer: The product is $(-1)^{n-1}$.

(b) Explain why conjugate pairing gives the same result, distinguishing even and odd n. [3 marks]

All roots have modulus 1, so $z\bar{z}=1$. The remaining real factors determine the sign.

Answer: Each non-real conjugate pair has product 1. For odd n the only real root is 1, so the product is 1; for even n the real roots are 1 and -1, so the product is -1.