Complex Numbers - Paper 4

10 questions | 59 marks | 70 minutes

Roots and factorisations, from recognising a square to exact root geometry and real coefficient constraints.

Equivalent correct methods earn the corresponding marks.

1. Let $z = 1 - \sqrt{3}i$. [5 marks]

(a) Write z in principal-argument polar form. [2 marks]

The modulus is 2 and the point is in quadrant IV.

Answer: $2\text{cis}(-\frac{\pi}{3})$.

(b) Write its reciprocal in polar and Cartesian forms. [3 marks]

Invert the modulus and negate the angle; then use exact trigonometric values.

Answer: $\frac{1}{2}\text{cis}(\frac{\pi}{3}) = \frac{1}{4} + \frac{\sqrt{3}}{4}i$.

2. A complex square is $3 + 4i$. [5 marks]

(a) Calculate $(2+i)^2$. [2 marks]

$4 + 4i + i^2 = 3 + 4i$.

Answer: $3 + 4i$.

(b) Hence solve $z^2 = 3 + 4i$ and explain why your list is complete. [3 marks]

Factor $z^2 - (2+i)^2 = (z - (2+i))(z + (2+i))$. A product is zero only if a factor is zero.

Answer: $z = 2 + i$ or $z = -2 - i$.

3. Find the cube roots of $-27i$. [6 marks]

(a) Give all roots with principal arguments. [3 marks]

Use modulus 3 and arguments $\frac{-\frac{\pi}{2}+2k\pi}{3}$ for $k=0,1,2$, reducing $\frac{7\pi}{6}$ to $-\frac{5\pi}{6}$.

Answer: $3\text{cis}(-\frac{\pi}{6}),\ 3\text{cis}(\frac{\pi}{2}),\ 3\text{cis}(-\frac{5\pi}{6})$.

(b) Convert the roots to Cartesian form and find their sum. [3 marks]

The real components cancel; the imaginary components sum to $-\frac{3}{2}+3-\frac{3}{2}=0$.

Answer: $\frac{3\sqrt{3}}{2}-\frac{3}{2}i,\ 3i,\ -\frac{3\sqrt{3}}{2}-\frac{3}{2}i$; sum 0.

4. Consider $z^5=-32$. [6 marks]

(a) Give a formula listing all five roots. [3 marks]

$-32=32\text{cis}\,\pi$; take fifth-root modulus 2 and divide all five relevant arguments by 5.

Answer: $z_k=2\text{cis}(\frac{\pi}{5}+\frac{2k\pi}{5}),\ k=0,1,2,3,4$.

(b) Identify the real root and state how many roots have positive real part. [3 marks]

The angles are $\frac{\pi}{5},\frac{3\pi}{5},\pi,\frac{7\pi}{5},\frac{9\pi}{5}$. Only the first and last have positive cosine.

Answer: Real root -2; two roots have positive real part.

5. The roots of $z^4=16i$ are $z_k=2\text{cis}(\frac{\pi}{8}+\frac{k\pi}{2})$ for $k=0,1,2,3$. [6 marks]

(a) Show that opposite roots sum to zero and hence find the sum of all roots. [3 marks]

The arguments of opposite roots differ by π, so one is the negative of the other.

Answer: $z_0+z_2=0$ and $z_1+z_3=0$; total sum 0.

(b) Calculate the product of all four roots in polar and Cartesian forms. [3 marks]

The product modulus is $2^4=16$. The angles sum to $4(\frac{\pi}{8})+(0+1+2+3)\frac{\pi}{2}=\frac{7\pi}{2}$.

Answer: $16\text{cis}(\frac{7\pi}{2})=-16i$.

6. Consider $P(z) = z^6 - 1$. [7 marks]

(a) List the six roots in polar form. [3 marks]

These are the sixth roots of unity, equally spaced by $\frac{\pi}{3}$.

Answer: $\text{cis}(\frac{k\pi}{3})$, $k = 0, 1, 2, 3, 4, 5$.

(b) Factorise P into real linear and irreducible real quadratic factors. [4 marks]

The real roots are ± 1. Pair the conjugate roots with real parts $\frac{1}{2}$ and $-\frac{1}{2}$; each pair has product 1.

Answer: $(z-1)(z+1)(z^2 - z + 1)(z^2 + z + 1)$.

7. Consider $P(z) = z^4 + 4$. [6 marks]

(a) Show that $P(z) = (z^2 - 2z + 2)(z^2 + 2z + 2)$. [3 marks]

It is $(z^2 + 2)^2 - (2z)^2 = z^4 + 4z^2 + 4 - 4z^2$.

Answer: The product is $z^4 + 4$.

(b) Find every complex root. [3 marks]

The first quadratic is $(z-1)^2 + 1$ and the second is $(z+1)^2 + 1$.

Answer: $1 + i, 1 - i, -1 + i, -1 - i$.

8. A monic real cubic has a root $1 + 2i$ and its coefficient of z^2 is -7. [6 marks]

(a) Find all its roots. [3 marks]

The conjugate root is forced. The roots sum to 7, so the third is $7 - 2 = 5$.

Answer: $1 + 2i, 1 - 2i, 5$.

(b) Find the cubic in expanded form. [3 marks]

Multiply $(z-5)(z^2 - 2z + 5)$ and collect terms.

Answer: $z^3 - 7z^2 + 15z - 25$.

9. The real polynomial $P(z) = (z^2 + 4)^2(z - 1)^2$ has degree 6. [6 marks]

(a) List its distinct roots and their multiplicities. [3 marks]

Each of $z - 2i$, $z + 2i$ and $z - 1$ occurs twice in the complete factorisation.

Answer: $2i,\ -2i, 1$, each of multiplicity 2.

(b) Explain how the root count agrees with the degree and with conjugate pairing. [3 marks]

Count repeated factors, rather than only the three distinct values.

Answer: The multiplicities total $2 + 2 + 2 = 6$. The roots $2i$ and $-2i$ form a pair with equal multiplicity, and 1 is real.

10. A designer places five equally spaced markers at the roots of $z^5 = 32$. [6 marks]

(a) Find the distance between neighbouring markers in exact trigonometric form. [3 marks]

The markers lie on a circle of radius 2 with central angle $\frac{2\pi}{5}$. Bisecting the isosceles triangle gives chord length $2(2)\sin(\frac{\pi}{5})$.

Answer: $4\sin(\frac{\pi}{5})$.

(b) An enlargement multiplies all positions by $3\text{cis}(\frac{\pi}{10})$. Give the new radius and the new first marker's argument if the original first marker was 2. [3 marks]

The multiplier triples all distances from the origin and rotates every position by $\frac{\pi}{10}$.

Answer: New radius 6; first-marker argument $\frac{\pi}{10}$.