Complex Numbers - Paper 3

Name: _____ Date: _____

10 questions | 56 marks | 70 minutes

Reasoning and proof: inspect claims, derive identities and connect algebra with geometry.

No calculator needed.

1. A student converts $-2+2i$ to polar form using $\tan^{-1}(-1)=-\frac{\pi}{4}$. [5 marks]

(a) Explain the quadrant error and give the correct principal-argument polar form. [3 marks]

(b) Give another argument outside the principal interval. [2 marks]

2. Let $z=x+iy$, where x and y are real. [5 marks]

(a) Prove $z\bar{z}=|z|^2$. [3 marks]

(b) Hence show that $\frac{1}{z}=\bar{z}$ when $|z|=1$. [2 marks]

3. Let $u = r\text{cis}\,\alpha$ and $v = s\text{cis}\,\beta$, where r and s are positive. [6 marks]

(a) Prove $|uv| = |u||v|$. [3 marks]

(b) Is $\text{Arg}(uv) = \text{Arg}\,u + \text{Arg}\,v$ always literally true? Give a counterexample under $(-\pi, \pi]$. [3 marks]

4. Let $c = \cos\theta$ and $s = \sin\theta$. [6 marks]

(a) Expand $(c + is)^3$ into real and imaginary components. [3 marks]

(b) Use de Moivre to prove $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3 marks]

5. A proposed solution of $z^3 = 1$ lists only $z = 1$, saying that the cube root of 1 is 1. [5 marks]

(a) Find the two omitted roots in Cartesian form. [3 marks]

(b) Explain the difference between one principal root and solving the equation. [2 marks]

6. Let $\omega = \text{cis}(\frac{2\pi}{3})$. [6 marks]

(a) Prove $1 + \omega + \omega^2 = 0$ without decimal approximations. [3 marks]

(b) Hence show that $(1 - \omega)(1 - \omega^2) = 3$. [3 marks]

7. Two points A and B represent distinct complex numbers a and b. Their midpoint represents $m = \frac{a+b}{2}$. [6 marks]

(a) Show that $|m - a| = |m - b|$. [3 marks]

(b) Does being equidistant from A and B force a point to be their midpoint? Give a concrete counterexample. [3 marks]

8. An instrument reports $\operatorname{Arg}(z - 2i) = 0$. [6 marks]

(a) Describe the locus with all endpoint restrictions. [3 marks]

(b) Find its intersection with $|z| = \sqrt{13}$. [3 marks]

9. A real polynomial P has the non-real root $a+bi$, where a and b are real and $b \neq 0$. [6 marks]

(a) Explain why the conjugate is a root. [3 marks]

(b) Find the real quadratic factor contributed by the pair. [3 marks]

10. A student claims that every real cubic has exactly one real root and two non-real roots. [5 marks]

(a) Give a real cubic that disproves the claim. [2 marks]

(b) State and justify the correct weaker conclusion for every real cubic. [3 marks]