Complex Numbers - Paper 2

10 questions | 55 marks | 70 minutes

A second balanced paper with signals, conjugates, roots, loci and reconstruction of polynomials.

Equivalent correct methods earn the corresponding marks.

1. Consider $z = 5\text{cis}(\frac{7\pi}{6})$. [4 marks]

(a) Write z in Cartesian form. [2 marks]

$\cos(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2}$ and $\sin(\frac{7\pi}{6}) = -\frac{1}{2}$.

Answer: $-\frac{5\sqrt{3}}{2} - \frac{5}{2}i$.

(b) Give its principal argument and the principal argument of its conjugate. [2 marks]

Subtract 2π for the first; conjugation reflects the point in the real axis.

Answer: $-\frac{5\pi}{6}$ and $\frac{5\pi}{6}$.

2. Let $z = 1 - 2i$. [6 marks]

(a) Calculate z^2 and $z\bar{z}$. [3 marks]

Expand the square; the conjugate product is $1^2 + (-2)^2 = 5$.

Answer: $z^2 = -3 - 4i$ and $z\bar{z} = 5$.

(b) Find $\frac{1}{z}$ and verify by multiplication. [3 marks]

$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{1+2i}{5}$; multiplying by $1 - 2i$ gives $\frac{5}{5} = 1$.

Answer: $\frac{1+2i}{5}$.

3. Two idealised same-frequency signals are represented by $A = 4\text{cis}(\frac{\pi}{3})$ and $B = 4\text{cis}(-\frac{\pi}{3})$. Their resultant is $A + B$. [5 marks]

(a) Find the resultant in Cartesian and polar forms. [3 marks]

The signals are $2 + 2\sqrt{3}i$ and $2 - 2\sqrt{3}i$; their imaginary components cancel.

Answer: $A + B = 4 = 4\text{cis}\,0$.

(b) Explain why the resultant amplitude is not 8. [2 marks]

Each has modulus 4, but the phase difference is $\frac{2\pi}{3}$. Only their real components reinforce here.

Answer: The signals point in different directions, so their moduli cannot simply be added.

4. The map $w = (1 - i)z$ acts on the Argand plane. [5 marks]

(a) Describe its scale factor and rotation. [2 marks]

$1 - i = \sqrt{2}\,\text{cis}(-\frac{\pi}{4})$.

Answer: Scale factor $\sqrt{2}$ and clockwise rotation through $\frac{\pi}{4}$.

(b) Find the original point if its image is $w = 4i$. [3 marks]

$z = \frac{4i}{1-i} = \frac{4i(1+i)}{2} = -2 + 2i$.

Answer: $z = -2 + 2i$.

5. Use de Moivre's theorem. [6 marks]

(a) Find $(\sqrt{3}+i)^6$. [3 marks]

$\sqrt{3}+i=2\text{cis}(\frac{\pi}{6})$, so the sixth power is $64\text{cis}\,\pi=-64$.

Answer: -64.

(b) Find the least positive integer n for which $(\text{cis}(\frac{3\pi}{5}))^n=1$. [3 marks]

Require $\frac{3n\pi}{5}=2k\pi$, hence $3n=10k$. The least positive integer solution has $n=10$.

Answer: $n=10$.

6. Solve $z^4=81$. [6 marks]

(a) List every root in Cartesian form. [3 marks]

The fourth-root modulus is 3; arguments are $0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$.

Answer: $3, 3i, -3, -3i$.

(b) Find the area of the square formed by these points. [3 marks]

The diagonals have length 6 and are perpendicular; the area is $\frac{1}{2}(6)(6)=18$.

Answer: 18 square units.

7. Let $\zeta=\text{cis}(\frac{2\pi}{5})$. [5 marks]

(a) Find ζ^{17} as a power from 0 to 4, and find $\overline{\zeta}$ as a power of ζ. [3 marks]

Reduce 17 modulo 5. Also $\overline{\zeta}=\zeta^{-1}=\zeta^4$.

Answer: $\zeta^{17}=\zeta^2$; $\overline{\zeta}=\zeta^4$.

(b) Find $1+\zeta+\zeta^2+\zeta^3$. [2 marks]

The sum of all five fifth roots is zero; subtract the omitted term.

Answer: $-\zeta^4$.

8. Consider the locus $|z-2|=|z+2i|$. [6 marks]

(a) Find its Cartesian equation. [3 marks]

For $z=x+iy$, $(x-2)^2+y^2=x^2+(y+2)^2$; cancelling yields $-4x=4y$.

Answer: $y=-x$.

(b) Find its intersections with $|z|=\sqrt{2}$. [3 marks]

Use $y=-x$ in $x^2+y^2=2$ to get $2x^2=2$, so $x=\pm1$.

Answer: $z=1-i$ and $z=-1+i$.

9. Let $P(z)=z^3+z^2+4z+4$. [6 marks]

(a) Show that -1 is a root and factorise over the real numbers. [3 marks]

$-1+1-4+4=0$. Group $z^2(z+1)+4(z+1)$.

Answer: $P(-1)=0$; $P(z)=(z+1)(z^2+4)$.

(b) State all roots and factorise completely over the complex numbers. [3 marks]

$z^2+4=0$ gives the two non-real roots.

Answer: Roots $-1, 2i, -2i$; $(z+1)(z-2i)(z+2i)$.

10. A monic real quartic has roots $1+i$ and $-1+2i$. [6 marks]

(a) Give its factorisation into real quadratic factors. [3 marks]

The other roots are $1-i$ and $-1-2i$; combine each conjugate pair.

Answer: $(z^2-2z+2)(z^2+2z+5)$.

(b) Find its constant coefficient and the sum of all its roots without fully expanding. [3 marks]

The constant is 2×5. The two conjugate-pair sums are 2 and -2.

Answer: Constant coefficient 10; sum of roots 0.