Mathematical induction - Practice test 6

10 questions | 56 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A proof establishes $P(5)$ and $P(k)$ implies $P(k+1)$ for every integer k at least 5. [4 marks]

(a) State the exact range established by the proof. [2 marks]

The starting case reaches all later cases by a one-step implication.

Answer: Every integer n at least 5.

(b) Explain whether $P(4)$ follows, and what is needed to claim all positive integers. [2 marks]

The established implication does not run backwards.

Answer: $P(4)$ does not follow. The earlier cases need separate proofs or a valid step covering them.

2. Prove $n^5 - n$ is divisible by 5 for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, the value is zero, a multiple of 5.

Assume $k^5 - k = 5q$ for an integer q.

Answer: At $n = 1$, the value is zero, a multiple of 5. Assume $k^5 - k = 5q$ for an integer q.

(b) Establish the successor and conclude the stated domain. [4 marks]

Expansion gives $(k+1)^5 - (k+1) = (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k$.

Factor 5 to obtain $5[q + k^4 + 2k^3 + 2k^2 + k]$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

3. Prove $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides are $\frac{1}{3}$.

Assume the sum to k is $\frac{k}{2k+1}$.

Answer: At $n = 1$, both sides are $\frac{1}{3}$. Assume the sum to k is $\frac{k}{2k+1}$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $\frac{1}{(2k+1)(2k+3)}$.

The numerator is $k(2k+3)+1 = (k+1)(2k+1)$.

Cancellation leaves $\frac{k+1}{2k+3}$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

4. Prove $\sum_{r=1}^{n} (1+2+\cdots+r) = \frac{n(n+1)(n+2)}{6}$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the total of the first k triangular rows is $\frac{k(k+1)(k+2)}{6}$.

Answer: At $n = 1$, both sides are 1. Assume the total of the first k triangular rows is $\frac{k(k+1)(k+2)}{6}$.

(b) Establish the successor and conclude the stated domain. [4 marks]

The next triangular row contributes $\frac{(k+1)(k+2)}{2}$.

Factor $(k+1)(k+2)$ to obtain $\frac{(k+1)(k+2)(k+3)}{6}$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

5. Prove $10^n - 3^n$ is divisible by 7 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 7.

Assume $10^k - 3^k = 7q$ for an integer q.

Answer: $n = 1$ gives 7. Assume $10^k - 3^k = 7q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $10^{k+1} - 3^{k+1} = 10(10^k - 3^k) + (10 - 3)3^k$.

This is $7(10q + 3^k)$, an integer multiple of 7.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. The first case has been checked for $7^n - (-1)^n$ is divisible by 8. Complete the remaining induction argument. [6 marks]

(a) State exactly what is assumed at an arbitrary kth case. [1 mark]

Assume $7^k - (-1)^k = 8q$ for an integer q.

Answer: Assume $7^k - (-1)^k = 8q$ for an integer q.

(b) Derive the successor case. Identify where the hypothesis is used. [4 marks]

$7^{k+1} - (-1)^{k+1} = 7[7^k - (-1)^k] + 8(-1)^k$.

This is $8[7q + (-1)^k]$, divisible by 8.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

(c) Explain why the checked base plus this implication covers all positive integers. [1 mark]

The chain starts at 1 and advances by one.

Answer: Every positive index is reached by successive one-step implications.

7. Let $W_n = \sum_{r=1}^{n} r^2 2^r$ for positive integers n. [6 marks]

(a) Check the base case and state the hypothesis for $W_n = (n^2 - 2n + 3)2^{n+1} - 6$. [2 marks]

The initial sum consists of the single term $1^2 2^1 = 2$.

Answer: $W_1 = 2 = (1 - 2 + 3)2^2 - 6$. Assume $W_k = (k^2 - 2k + 3)2^{k+1} - 6$ for arbitrary integer k at least 1.

(b) Prove the successor formula and hence calculate W_4. [4 marks]

Add the new term $(k+1)^2 2^{k+1}$.

Factoring 2^{k+1} gives $[k^2 - 2k + 3 + (k+1)^2]2^{k+1} - 6 = (k^2 + 2)2^{k+2} - 6$.

Since $(k+1)^2 - 2(k+1) + 3 = k^2 + 2$, this is the formula at $k+1$. Conclude by induction.

$W_4 = (16 - 8 + 3)2^5 - 6 = 346$.

Answer: The formula holds throughout the stated domain and $W_4 = 346$.

8. Consider $R_n = \sum_{r=1}^{n} \frac{r^2}{4r^2 - 1}$ for positive integers n. [6 marks]

(a) Verify the first case of $R_n = \frac{n(n+1)}{2(2n+1)}$, and state the induction hypothesis and new term. [2 marks]

Factor $4(k+1)^2 - 1$ as $(2k+1)(2k+3)$.

Answer: $R_1 = \frac{1}{3}$; assume $R_k = \frac{k(k+1)}{2(2k+1)}$. The new term is $\frac{(k+1)^2}{(2k+1)(2k+3)}$.

(b) Establish the successor case and conclude. [4 marks]

The next sum over $2(2k+1)(2k+3)$ has numerator $k(k+1)(2k+3) + 2(k+1)^2$.

Factor it as $(k+1)[2k^2 + 5k + 2] = (k+1)(2k+1)(k+2)$.

Cancel the nonzero $2k+1$ to obtain $\frac{(k+1)(k+2)}{2(2k+3)}$. The base and implication prove the formula.

Answer: $R_n = \frac{n(n+1)}{2(2n+1)}$ for all positive integers n.

9. A computer checked $\prod_{r=1}^{n} \frac{2r-1}{2r+1} = \frac{1}{2n+1}$ for the first one hundred positive integers. Replace that numerical evidence with a proof. [6 marks]

(a) Explain the limitation of the computer check and verify a valid starting case. [2 marks]

At $n = 1$, the product is $\frac{1}{3}$.

An arbitrary successor implication remains necessary.

Answer: At $n = 1$, the product is $\frac{1}{3}$. Finite checks do not establish untested cases.

(b) State the hypothesis and derive the next case with exact algebra. [4 marks]

Assume the product to k is $\frac{1}{2k+1}$.

Multiply by $\frac{2k+1}{2k+3}$.

The result is $\frac{1}{2k+3} = \frac{1}{2(k+1)+1}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

10. A divisibility proof concludes $A(k+1) = d(q+\frac{1}{2})$ with integer q. Explain the problem. [4 marks]

(a) Explain why $q+\frac{1}{2}$ is not an integer multiplier and why this fails to establish divisibility. [2 marks]

Integer divisibility requires an integer multiplier.

An arbitrary rational multiplier does not suffice.

Answer: The proposed multiplier is not an integer.

(b) State the required form of the conclusion. [2 marks]

A non-integer multiplier does not express divisibility.

Answer: $A(k+1) = dt$ with t an integer.