Mathematical induction - Practice test 5

10 questions | 57 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A triangular stack has n rows; row r contains exactly 2r tiles. [5 marks]

(a) Find the total tile count through row n. [2 marks]

It is twice $1+2+\cdots+n$.

Answer: $n(n+1)$.

(b) If the total formula holds through row k, write the next row count and obtain the next total. [3 marks]

$k(k+1)+2(k+1)=(k+1)(k+2)$.

Answer: The next row has $2(k+1)$ tiles, giving $(k+1)(k+2)$ total.

2. Prove $\sum_{r=1}^{n} r(r+1)(r+2)=\frac{n(n+1)(n+2)(n+3)}{4}$ for integers $n\geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n=1$, both sides equal 6.

Assume the formula is true for the sum ending at k.

Answer: At $n=1$, both sides equal 6. Assume the formula is true for the sum ending at k.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $(k+1)(k+2)(k+3)$.

Factor the common product to get $\frac{(k+1)(k+2)(k+3)(k+4)}{4}$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

3. Prove $\prod_{r=1}^{n} \frac{r+2}{r} = \frac{(n+1)(n+2)}{2}$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides equal 3.

Assume the product to k is $\frac{(k+1)(k+2)}{2}$.

Answer: At $n = 1$, both sides equal 3. Assume the product to k is $\frac{(k+1)(k+2)}{2}$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Multiply by the next factor $\frac{k+3}{k+1}$.

Cancel the nonzero $k+1$ to obtain $\frac{(k+2)(k+3)}{2}$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

4. Prove $4^n + 6n - 1$ is divisible by 9 for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, the value is 9.

Assume $A_k = 4^k + 6k - 1 = 9q$ for an integer q.

Answer: At $n = 1$, the value is 9. Assume $A_k = 4^k + 6k - 1 = 9q$ for an integer q.

(b) Establish the successor and conclude the stated domain. [4 marks]

$A_{k+1} = 4A_k + 9(1 - 2k)$.

The right side is $9(4q + 1 - 2k)$, an integer multiple of 9.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

5. Prove $11^n - 3^n$ is divisible by 8 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 8.

Assume $11^k - 3^k = 8q$ for an integer q.

Answer: $n = 1$ gives 8. Assume $11^k - 3^k = 8q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $11^{k+1} - 3^{k+1} = 11(11^k - 3^k) + (11 - 3)3^k$.

This is $8(11q + 3^k)$, an integer multiple of 8.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Consider the alternating sum $S_n = \sum_{r=1}^{n} (-1)^{r-1} r^2$. [6 marks]

(a) Calculate S_1, S_2 and S_3, then check the formula $S_n = \frac{(-1)^{n-1} n(n+1)}{2}$ at the first case. [2 marks]

The signs alternate: $1 - 4 + 9 = 6$.

Answer: $S_1 = 1$, $S_2 = -3$, $S_3 = 6$. The first-case formula is 1.

(b) Prove the formula by induction, taking care with the sign of the new term. [4 marks]

Assume $S_k = \frac{(-1)^{k-1} k(k+1)}{2}$. The new term is $(-1)^k (k+1)^2$.

Using $(-1)^{k-1} = -(-1)^k$ gives $S_{k+1} = (-1)^k (k+1)[-\frac{k}{2} + (k+1)]$.

This simplifies to $\frac{(-1)^k (k+1)(k+2)}{2}$, the formula at $k+1$. The base and step prove the claim.

Answer: $S_n = \frac{(-1)^{n-1} n(n+1)}{2}$ for all positive integers n.

7. The first case has been checked for $\sum_{r=1}^{n}(2r-1)^2=\frac{n(4n^2-1)}{3}$. Complete the remaining induction argument. [6 marks]

(a) State exactly what is assumed at an arbitrary kth case. [1 mark]

Assume the formula holds at $n=k$.

Answer: Assume the formula holds at $n=k$.

(b) Derive the successor case. Identify where the hypothesis is used. [4 marks]

Add $(2k+1)^2$.

The numerator becomes $4k^3+12k^2+11k+3=(k+1)(4k^2+8k+3)$.

This equals $(k+1)[4(k+1)^2-1]$, as required.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

(c) Explain why the checked base plus this implication covers all positive integers. [1 mark]

The chain starts at 1 and advances by one.

Answer: Every positive index is reached by successive one-step implications.

8. For positive integers n, let $D_n=\sum_{r=1}^{n}\frac{r}{2^r}$. [6 marks]

(a) Verify the proposed formula $D_n=2-\frac{n+2}{2^n}$ at $n=1$, and state an exact induction hypothesis. [2 marks]

The same arbitrary k must appear in the upper bound and both places on the right.

Answer: $D_1=\frac{1}{2}=2-\frac{3}{2}$. Assume $D_k=2-\frac{k+2}{2^k}$ for an arbitrary integer k at least 1.

(b) Prove the successor and use the formula to calculate D_6 exactly. [4 marks]

Add $\frac{k+1}{2^{k+1}}$. The next sum is $2-\frac{2(k+2)-(k+1)}{2^{k+1}}=2-\frac{k+3}{2^{k+1}}$.

This is the required successor, establishing the formula by induction.

$D_6=2-\frac{8}{64}=\frac{15}{8}$.

Answer: The formula holds for all positive integers n, and $D_6=\frac{15}{8}$.

9. For positive integers n, define the product of the first n odd numbers by $O_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$. [6 marks]

(a) Explain the identity $(2n)! = 2^n n!\, O_n$ by grouping the even and odd factors. [2 marks]

There are n even factors $2, 4, \ldots, 2n$, whose product is $2^n(1 \cdot 2 \cdots n)$. The remaining factors are exactly those in O_n.

Answer: The even factors multiply to $2^n n!$, and the odd factors multiply to O_n.

(b) Independently prove $O_n = \frac{(2n)!}{2^n n!}$ by induction. [4 marks]

At $n = 1$, $1 = \frac{2!}{2 \cdot 1!}$. Assume the stated formula at k.

The next product is $\frac{(2k+1)(2k)!}{2^k k!}$.

The target is $\frac{(2k+2)!}{2^{k+1}(k+1)!}$. Expanding the factorials cancels $\frac{2k+2}{2(k+1)} = 1$ and leaves exactly the next product.

The first case and implication prove the formula.

Answer: The formula is true for every positive integer n.

10. A sum of odd numbers proof adds $2k-1$ to the assumed total k^2. Correct the next term. [4 marks]

(a) Identify the next odd term that must be added and explain why $2k-1$ repeats an old term. [2 marks]

The assumed sum already contains $2k-1$.

The next odd number is $2(k+1)-1$.

Answer: It must add $2k+1$.

(b) Show the corrected successor total. [2 marks]

Expand the square on the right to verify the bridge.

Answer: $k^2 + (2k+1) = (k+1)^2$.