Mathematical induction - Practice test 4

Name: _____ Date: _____

10 questions | 58 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

No calculator needed.

1. A proposed identity is $\sum_{r=1}^{n}(2r+1)=n^2$. [5 marks]

(a) Test the first case and decide whether induction can prove this identity. [2 marks]

(b) Find and justify a corrected formula. [3 marks]

2. Prove $\sum_{r=1}^{n}(5r-2)=\frac{5n(n+1)}{2}-2n$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

(b) Prove the next case and conclude. [4 marks]

3. Prove $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

(b) Establish the successor and conclude the stated domain. [4 marks]

4. Prove $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

(b) Establish the successor and conclude the stated domain. [4 marks]

5. Prove $(1-x)(1+x+\cdots+x^n)=1-x^{n+1}$ for integers $n \geq 1$ and every real x. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

(b) Establish the successor and conclude the stated domain. [4 marks]

6. Set $A_n = n^4 + 2n^3 - n^2 - 2n$ for positive integers n. [6 marks]

(a) Show $A_{k+1} - A_k = 4k(k+1)(k+2)$ and explain why this difference is divisible by 24. [3 marks]

(b) Use induction to prove that 24 divides A_n throughout the stated domain. [3 marks]

7. A display has n sections. Section r contains $(2r-1)^3$ illuminated points. [6 marks]

(a) Write the first three section sizes and verify the total formula $T_n = n^2(2n^2-1)$ at $n=1$. [2 marks]

(b) Prove the total formula by induction. [4 marks]

8. A proposed formula has the form $\sum_{r=1}^{n} r(r+3) = n(n+1)(an+b)$, with constants a and b. [6 marks]

(a) Use $n=1$ and $n=2$ to determine a and b. Explain why this alone is not a proof. [2 marks]

(b) Prove the resulting formula for all positive integers n. [4 marks]

9. The first case has been checked for $\prod_{r=1}^{n} \frac{2r-1}{2r+1} = \frac{1}{2n+1}$. Complete the remaining induction argument. [6 marks]

(a) State exactly what is assumed at an arbitrary kth case. [1 mark]

(b) Derive the successor case. Identify where the hypothesis is used. [4 marks]

(c) Explain why the checked base plus this implication covers all positive integers. [1 mark]

10. Refute the claim that $n^2 + n + 41$ is prime for every positive integer n. [5 marks]

(a) Give a positive integer n for which the expression is composite, and show a non-trivial factorisation. [3 marks]

(b) Explain why one such counterexample is sufficient. [2 marks]