Mathematical induction - Practice test 4

10 questions | 58 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A proposed identity is $\sum_{r=1}^{n}(2r+1)=n^2$. [5 marks]

(a) Test the first case and decide whether induction can prove this identity. [2 marks]

A false allowed case prevents a valid proof of the claim.

Answer: At $n=1$, the sum is 3 but the proposed value is 1, so the statement is false.

(b) Find and justify a corrected formula. [3 marks]

Summing $2r+1$ gives $2[\frac{n(n+1)}{2}]+n=n(n+2)$.

Answer: $n(n+2)$.

2. Prove $\sum_{r=1}^{n}(5r-2)=\frac{5n(n+1)}{2}-2n$ for all integers $n\geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives 3 on both sides.

Assume the formula holds at $n=k$.

Answer: $n=1$ gives 3 on both sides. Assume the formula holds at $n=k$.

(b) Prove the next case and conclude. [4 marks]

Add $5(k+1)-2$.

Grouping gives $\frac{5(k+1)(k+2)}{2}-2(k+1)$, the required next formula.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides are $\frac{1}{3}$.

Assume the stated expression holds for the sum to k.

Answer: At $n = 1$, both sides are $\frac{1}{3}$. Assume the stated expression holds for the sum to k.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $\frac{1}{(k+1)(k+3)}$.

Over $4(k+1)(k+2)(k+3)$, the numerator is $k(3k+5)(k+3)+4(k+2)$.

Factor it as $(k+1)^2(3k+8)$ and cancel one factor $k+1$.

The result is $\frac{(k+1)(3k+8)}{4(k+2)(k+3)}$, exactly the next formula.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

4. Prove $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, both sides are $\frac{1}{2}$.

Assume the sum to k is $1 - \frac{1}{(k+1)!}$.

Answer: At $n = 1$, both sides are $\frac{1}{2}$. Assume the sum to k is $1 - \frac{1}{(k+1)!}$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $\frac{k+1}{(k+2)!}$.

Since $\frac{1}{(k+1)!} = \frac{k+2}{(k+2)!}$, the difference becomes $1 - \frac{1}{(k+2)!}$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

5. Prove $(1-x)(1+x+\cdots+x^n)=1-x^{n+1}$ for integers $n\geq 1$ and every real x. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n=1$, $(1-x)(1+x)=1-x^2$.

Assume the identity through the power x^k, for an arbitrary real x. Keep x fixed through the induction step.

Answer: At $n=1$, $(1-x)(1+x)=1-x^2$. Assume the identity through the power x^k, for an arbitrary real x. Keep x fixed through the induction step.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $(1-x)x^{k+1}$ to the assumed left side.

The right side becomes $1-x^{k+1}+(1-x)x^{k+1}=1-x^{k+2}$.

No division by $1-x$ was used, so $x=1$ is included.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

6. Set $A_n=n^4+2n^3-n^2-2n$ for positive integers n. [6 marks]

(a) Show $A_{k+1}-A_k=4k(k+1)(k+2)$ and explain why this difference is divisible by 24. [3 marks]

Expanding and subtracting gives $4k^3+12k^2+8k=4k(k+1)(k+2)$.

Among three consecutive integers there is a multiple of 3 and at least one even integer. Their product is a multiple of 6. Multiplication by 4 makes a multiple of 24.

Answer: $A_{k+1}-A_k=4k(k+1)(k+2)$, a multiple of 24.

(b) Use induction to prove that 24 divides A_n throughout the stated domain. [3 marks]

$A_1=0=24(0)$. Assume $A_k=24q$ for an integer q.

By part (a), $4k(k+1)(k+2)=24t$ for an integer t, so $A_{k+1}=24(q+t)$.

The base and implication establish the claim.

Answer: 24 divides A_n for every positive integer n.

7. A display has n sections. Section r contains $(2r-1)^3$ illuminated points. [6 marks]

(a) Write the first three section sizes and verify the total formula $T_n = n^2(2n^2-1)$ at $n=1$. [2 marks]

The section rule uses odd-number cubes. The total includes every section from 1 to n.

Answer: The sizes are 1, 27 and 125. At $n=1$, $T_1 = 1^2(2-1) = 1$.

(b) Prove the total formula by induction. [4 marks]

Assume $T_k = k^2(2k^2-1)$ for arbitrary positive integer k.

The new section has $(2k+1)^3$ points. Thus $T_{k+1} = 2k^4 - k^2 + (2k+1)^3$.

Expanding gives $2k^4 + 8k^3 + 11k^2 + 6k + 1 = (k+1)^2[2(k+1)^2 - 1]$.

The checked base and successor implication prove the rule.

Answer: $T_n = n^2(2n^2-1)$ for all positive integers n.

8. A proposed formula has the form $\sum_{r=1}^{n} r(r+3) = n(n+1)(an+b)$, with constants a and b. [6 marks]

(a) Use $n=1$ and $n=2$ to determine a and b. Explain why this alone is not a proof. [2 marks]

At $n=1$, $4 = 2(a+b)$, so $a+b=2$. At $n=2$, $14 = 6(2a+b)$, so $2a+b=\frac{7}{3}$.

Subtraction gives $a=\frac{1}{3}$, then $b=\frac{5}{3}$.

Answer: $a=\frac{1}{3}$, $b=\frac{5}{3}$. Two checked cases do not establish every positive integer case.

(b) Prove the resulting formula for all positive integers n. [4 marks]

The first case is already checked. Assume the sum to k is $\frac{k(k+1)(k+5)}{3}$.

Add $(k+1)(k+4)$ and factor $\frac{k+1}{3}$.

The remaining polynomial is $k(k+5) + 3(k+4) = k^2 + 8k + 12 = (k+2)(k+6)$.

The next sum is $\frac{(k+1)(k+2)(k+6)}{3}$, as required. Conclude by induction.

Answer: $\sum_{r=1}^{n} r(r+3) = \frac{n(n+1)(n+5)}{3}$.

9. The first case has been checked for $\prod_{r=1}^{n} \frac{2r-1}{2r+1} = \frac{1}{2n+1}$. Complete the remaining induction argument. [6 marks]

(a) State exactly what is assumed at an arbitrary kth case. [1 mark]

Assume the product to k is $\frac{1}{2k+1}$.

Answer: Assume the product to k is $\frac{1}{2k+1}$.

(b) Derive the successor case. Identify where the hypothesis is used. [4 marks]

Multiply by $\frac{2k+1}{2k+3}$.

The result is $\frac{1}{2k+3} = \frac{1}{2(k+1)+1}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

(c) Explain why the checked base plus this implication covers all positive integers. [1 mark]

The chain starts at 1 and advances by one.

Answer: Every positive index is reached by successive one-step implications.

10. Refute the claim that $n^2 + n + 41$ is prime for every positive integer n. [5 marks]

(a) Give a positive integer n for which the expression is composite, and show a non-trivial factorisation. [3 marks]

This allowed case is composite.

One counterexample refutes a universal claim.

Answer: $n = 41$ gives $41^2 + 41 + 41 = 41(43)$.

(b) Explain why one such counterexample is sufficient. [2 marks]

The allowed composite case already contradicts that assertion.

Answer: A universal assertion requires every allowed case to be true.