Mathematical induction - Practice test 3

10 questions | 57 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. The numbers $n(n+1)$ are even for positive integers n. [4 marks]

(a) Give a direct argument. [2 marks]

Their product is therefore an integer multiple of two.

Answer: One of two consecutive integers is even.

(b) Explain why a correct direct proof does not also need induction. [2 marks]

Induction is one proof method, not a mandatory extra check after a general direct argument.

Answer: It already treats every allowed n at once.

2. Prove $\sum_{r=1}^{2n}(-1)^{r-1}r=-n$ for integers $n\geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n=1,\ 1-2=-1$.

Assume the alternating sum through $2k$ is $-k$.

Answer: At $n=1,\ 1-2=-1$. Assume the alternating sum through $2k$ is $-k$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add the two new terms $(2k+1)-(2k+2)=-1$.

The next paired sum is $-k-1=-(k+1)$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

3. Prove $\sum_{r=1}^{n} r\,r! = (n+1)! - 1$ for integers $n \geq 1$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 1$, $1 = 2! - 1$.

Assume the sum to k equals $(k+1)! - 1$.

Answer: At $n = 1$, $1 = 2! - 1$. Assume the sum to k equals $(k+1)! - 1$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Add $(k+1)(k+1)!$.

The sum is $(k+1)!\,[1 + (k+1)] - 1 = (k+2)! - 1$.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

4. Prove $\prod_{r=2}^{n} \frac{r^2}{r^2 - 1} = \frac{2n}{n+1}$ for integers $n \geq 2$. [6 marks]

(a) Verify the first case and state a precise induction hypothesis. [2 marks]

At $n = 2$, $\frac{4}{3} = \frac{2(2)}{3}$.

Assume the product to k is $\frac{2k}{k+1}$, with $k \geq 2$.

Answer: At $n = 2$, $\frac{4}{3} = \frac{2(2)}{3}$. Assume the product to k is $\frac{2k}{k+1}$, with $k \geq 2$.

(b) Establish the successor and conclude the stated domain. [4 marks]

Multiply by $\frac{(k+1)^2}{(k+1)^2 - 1} = \frac{(k+1)^2}{k(k+2)}$.

The result is $\frac{2(k+1)}{k+2}$.

All denominators are nonzero because every product index is at least 2.

The first case and successor implication prove the entire stated domain by induction.

Answer: The identity holds throughout the stated domain.

5. Prove $7^n - 1$ is divisible by 6 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives 6.

Assume $7^k - 1 = 6q$ for an integer q.

Answer: $n = 1$ gives 6. Assume $7^k - 1 = 6q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $7^{k+1} - 1 = 7(7^k - 1) + 6$.

This is $6(7q + 1)$, an integer multiple of 6.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $7^n - (-1)^n$ is divisible by 8 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is 8.

Assume $7^k - (-1)^k = 8q$ for an integer q.

Answer: At $n = 1$, the value is 8. Assume $7^k - (-1)^k = 8q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

$7^{k+1} - (-1)^{k+1} = 7[7^k - (-1)^k] + 8(-1)^k$.

This is $8[7q + (-1)^k]$, divisible by 8.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

7. A student uses $S_k = \frac{k}{k+1}$ for $S_n = \sum_{r=1}^{n} \frac{1}{r(r+1)}$, then writes $S_{k+1} = S_k + \frac{1}{k(k+1)}$. [6 marks]

(a) Identify the error and write the correct successor expression. [2 marks]

The new index is $r = k+1$, which must replace r in both denominator factors.

Answer: The old last term has been added again. Use $S_{k+1} = S_k + \frac{1}{(k+1)(k+2)}$.

(b) Complete the corrected successor algebra, and give the exact value of the sum from $r = 2$ to $r = n$, for n at least 2. [4 marks]

$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

The first term is $\frac{1}{2}$, so subtract it: $\frac{n}{n+1} - \frac{1}{2} = \frac{n-1}{2(n+1)}$.

Answer: $S_{k+1} = \frac{k+1}{k+2}$; the sum from $r = 2$ is $\frac{n-1}{2(n+1)}$.

8. Prove $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ for every positive integer n. [6 marks]

(a) Check the base case and state the full induction hypothesis. [2 marks]

$1(3+1) = 1(1+1)^2 = 4$.

The hypothesis concerns the sum ending at k.

Answer: At $n = 1$, both sides are 4. Assume $\sum_{r=1}^{k} r(3r+1) = k(k+1)^2$ for an arbitrary integer k at least 1.

(b) Prove the successor case and conclude. [4 marks]

Add $(k+1)[3(k+1)+1] = (k+1)(3k+4)$.

The next sum is $(k+1)[k(k+1)+3k+4] = (k+1)(k+2)^2$.

This is the target formula at $k+1$, so the base case and implication prove the full domain.

Answer: The identity holds for all positive integers.

9. Let $P_n = \prod_{r=1}^{n} \frac{r(r+2)}{(r+1)^2}$ for positive integers n. [6 marks]

(a) Calculate P_1 and P_2, and conjecture a formula for P_n. [2 marks]

$P_2 = (\frac{3}{4})(\frac{8}{9}) = \frac{2}{3}$.

The displayed conjecture will be justified by induction, rather than by the two values alone.

Answer: $P_1 = \frac{3}{4}$, $P_2 = \frac{2}{3}$; a formula is $P_n = \frac{n+2}{2(n+1)}$.

(b) Prove the conjecture by induction, identifying the new factor and any denominator restrictions. [4 marks]

The first value matches. Assume $P_k = \frac{k+2}{2(k+1)}$.

Multiply by $\frac{(k+1)(k+3)}{(k+2)^2}$. The result is $\frac{k+3}{2(k+2)}$, the required value at $k+1$.

Every denominator is nonzero for positive integer indices. The base case and successor implication prove the formula.

Answer: $P_n = \frac{n+2}{2(n+1)}$ for all positive integers n.

10. The step is proved only for k at least 5 and the student checks $P(3)$. What is missing to establish every n at least 3? [5 marks]

(a) Identify the missing cases needed before the valid successor chain can cover every integer $n \geq 3$. [3 marks]

The implication cannot be used at $k = 3$ or $k = 4$.

$P(5)$ is needed to start the valid chain.

Answer: $P(4)$ and $P(5)$.

(b) If the missing cases are also proved, explain why the entire claimed domain follows. [2 marks]

Together these cover every integer at least 3.

Answer: $P(3)$ and $P(4)$ are individually known; $P(5)$ starts the valid chain through all larger indices.