# Mathematical induction - Practice test 2

10 questions | 57 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

**1.** A student wants to prove a formula for every integer $n$ at least 2. [5 marks]

**(a)** State the required starting case and explain the role of an arbitrary $k$. [2 marks]

The implication must work at every allowed stage, not one fixed integer.

**Answer:** Check $P(2)$, then use arbitrary $k$ at least 2.

**(b)** A proposed implication moves from $k$ to $k+2$. Give enough starting cases and explain why. [3 marks]

A step of two cannot change parity.

**Answer:** $P(2)$ and $P(3)$, one for each parity chain.

**2.** Prove $\sum_{r=1}^{n}(3r-1)=\frac{3n(n+1)}{2}-n$ for all integers $n\geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives $2$ on both sides.

Assume the formula holds at $n=k$.

**Answer:** $n=1$ gives $2$ on both sides. Assume the formula holds at $n=k$.

**(b)** Prove the next case and conclude. [4 marks]

Add $3(k+1)-1$.

Grouping gives $\frac{3(k+1)(k+2)}{2}-(k+1)$, the required next formula.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**3.** Prove $\sum_{r=1}^{n} r^3 = [\frac{n(n+1)}{2}]^2$ for all integers $n \geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are $1$.

Assume the sum to $k$ is $\frac{k^2(k+1)^2}{4}$.

**Answer:** At $n = 1$, both sides are $1$. Assume the sum to $k$ is $\frac{k^2(k+1)^2}{4}$.

**(b)** Prove the next case and conclude. [4 marks]

Add $(k+1)^3$ and factor $\frac{(k+1)^2}{4}$.

The remaining factor is $k^2 + 4k + 4 = (k+2)^2$.

The result is $[\frac{(k+1)(k+2)}{2}]^2$.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**4.** Prove $1 + 2 + 2^2 + \cdots + 2^{n-1} = (2^n - 1)$ for all integers $n \geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives $1 = 1$.

Assume the sum to $k$ is $(2^k - 1)$.

**Answer:** $n = 1$ gives $1 = 1$. Assume the sum to $k$ is $(2^k - 1)$.

**(b)** Prove the next case and conclude. [4 marks]

Add $2^k$.

The new total is $2^k - 1 + 2^k = 2^{k+1} - 1$.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**5.** Prove $8^n - 3^n$ is divisible by $5$ for all integers $n \geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

$n = 1$ gives $5$.

Assume $8^k - 3^k = 5q$ for an integer $q$.

**Answer:** $n = 1$ gives $5$. Assume $8^k - 3^k = 5q$ for an integer $q$.

**(b)** Prove the next case and conclude. [4 marks]

Write $8^{k+1} - 3^{k+1} = 8(8^k - 3^k) + (8 - 3)3^k$.

This is $5(8q + 3^k)$, an integer multiple of $5$.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**6.** Prove $n^3 + 2n$ is divisible by $3$ for all integers $n \geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is $3$.

Assume $k^3 + 2k = 3q$ for an integer $q$.

**Answer:** At $n = 1$, the value is $3$. Assume $k^3 + 2k = 3q$ for an integer $q$.

**(b)** Prove the next case and conclude. [4 marks]

$(k + 1)^3 + 2(k + 1) = (k^3 + 2k) + 3(k^2 + k + 1)$.

This is $3(q + k^2 + k + 1)$, an integer multiple of $3$.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**7.** Prove $\sum_{r=1}^{n} r\,2^{r-1} = (n-1)2^n + 1$ for all integers $n \geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are $1$.

Assume the sum to $k$ is $(k-1)2^k + 1$.

**Answer:** At $n = 1$, both sides are $1$. Assume the sum to $k$ is $(k-1)2^k + 1$.

**(b)** Prove the next case and conclude. [4 marks]

Add $(k+1)2^k$.

The power terms combine to $2k\,2^k = k2^{k+1}$, giving the required formula.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**8.** Prove $\prod_{r=1}^{n} \frac{2r-1}{2r+1} = \frac{1}{2n+1}$ for all integers $n \geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the product is $\frac{1}{3}$.

Assume the product to $k$ is $\frac{1}{2k+1}$.

**Answer:** At $n = 1$, the product is $\frac{1}{3}$. Assume the product to $k$ is $\frac{1}{2k+1}$.

**(b)** Prove the next case and conclude. [4 marks]

Multiply by $\frac{2k+1}{2k+3}$.

The result is $\frac{1}{2k+3} = \frac{1}{2(k+1)+1}$.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**9.** Prove $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all integers $n \geq 1$. [6 marks]

**(a)** Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $2 = \frac{1(2)(3)}{3}$.

Assume the formula holds at $n = k$.

**Answer:** At $n = 1$, $2 = \frac{1(2)(3)}{3}$. Assume the formula holds at $n = k$.

**(b)** Prove the next case and conclude. [4 marks]

Add $(k+1)(k+2)$.

Factoring gives $\frac{(k+1)(k+2)(k+3)}{3}$.

The true first case and successor implication establish the full domain by mathematical induction.

**Answer:** The statement holds for all positive integers.

**10.** Why do correct checks for the first one hundred positive integers not constitute induction? [4 marks]

**(a)** Explain why checking finitely many cases cannot establish the successor implication for an arbitrary positive integer. [2 marks]

An untested case could still fail.

A starting case and a general successor implication are both required.

**Answer:** There is no implication proved for an arbitrary integer.

**(b)** Which induction obligation has already been met, and which remains? [2 marks]

The first hundred checked cases include $n = 1$.

They do not establish $P(k)$ implies $P(k+1)$ for arbitrary positive $k$.

**Answer:** The base case $n = 1$ has been checked. A successor implication for every arbitrary positive integer $k$ remains to be proved.