Mathematical induction - Practice test 1

10 questions | 58 marks | 75 minutes

Proof structure, distinct summation identities, integer divisibility and criticism of arguments.

Equivalent correct methods earn the corresponding marks.

1. A paving design starts with four blocks. Each successive stage contains exactly three more blocks than the preceding stage. [5 marks]

(a) Find a rule for the number in stage n. [2 marks]

The construction gives an arithmetic pattern with first value 4 and difference 3.

Answer: $3n+1$.

(b) Suggest the cumulative total through stage n, and explain how it could be proved. [3 marks]

Sum the stage rule from 1 to n.

Check $n=1$, assume the kth total, then add the $(k+1)$th stage.

Answer: $T_n=\frac{3n(n+1)}{2}+n$.

2. Prove $1+2+\cdots+n=\frac{n(n+1)}{2}$ for all integers $n\geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n=1$, both sides are 1.

Assume the sum to arbitrary integer $k\geq 1$ is $\frac{k(k+1)}{2}$.

Answer: At $n=1$, both sides are 1. Assume the sum to arbitrary integer $k\geq 1$ is $\frac{k(k+1)}{2}$.

(b) Prove the next case and conclude. [4 marks]

Add the next term $k+1$.

The next sum is $\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $1+3+\cdots+(2n-1)=n^2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n=1$, $1=1^2$.

Assume the first k odd numbers sum to k^2.

Answer: At $n=1$, $1=1^2$. Assume the first k odd numbers sum to k^2.

(b) Prove the next case and conclude. [4 marks]

Add the next odd term $2k+1$.

The result is $k^2+2k+1=(k+1)^2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n=1$, $1=\frac{1(2)(3)}{6}$.

Assume the formula holds at arbitrary $n=k$.

Answer: At $n=1$, $1=\frac{1(2)(3)}{6}$. Assume the formula holds at arbitrary $n=k$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)^2$ and factor $\frac{k+1}{6}$.

The remaining factor is $k(2k+1)+6(k+1)=2k^2+7k+6$.

Factor it as $(k+2)(2k+3)$, giving the required formula at $k+1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $1+3+3^2+\cdots+3^{n-1}=\frac{3^n-1}{2}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives $\frac{3-1}{2}=1$.

Assume the sum to k is $\frac{3^k-1}{2}$.

Answer: $n=1$ gives $\frac{3-1}{2}=1$. Assume the sum to k is $\frac{3^k-1}{2}$.

(b) Prove the next case and conclude. [4 marks]

Add 3^k.

The common numerator is $3^k-1+2(3^k)=3^{k+1}-1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove 5^n-2^n is divisible by 3 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives 3.

Assume $5^k-2^k=3q$ for an integer q.

Answer: $n=1$ gives 3. Assume $5^k-2^k=3q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

Write $5^{k+1}-2^{k+1}=5(5^k-2^k)+(5-2)2^k$.

This is $3(5q+2^k)$, an integer multiple of 3.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

7. Prove $n^3 - n$ is divisible by 6 for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the value is $0 = 6(0)$.

Assume $k^3 - k = 6q$ for an integer q.

Answer: At $n = 1$, the value is $0 = 6(0)$. Assume $k^3 - k = 6q$ for an integer q.

(b) Prove the next case and conclude. [4 marks]

$(k+1)^3 - (k+1) = (k^3 - k) + 3k(k+1)$.

One of consecutive integers $k, k+1$ is even, so $k(k+1) = 2t$ for an integer t.

The next value is $6(q+t)$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

8. Prove $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are $\frac{1}{2}$.

Assume the sum to k is $\frac{k}{k+1}$.

Answer: At $n = 1$, both sides are $\frac{1}{2}$. Assume the sum to k is $\frac{k}{k+1}$.

(b) Prove the next case and conclude. [4 marks]

Add $\frac{1}{(k+1)(k+2)}$.

The numerator over the common denominator is $k(k+2) + 1 = (k+1)^2$.

Cancellation gives $\frac{k+1}{k+2}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

9. Prove $\prod_{r=1}^{n} \frac{r+1}{r} = n+1$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n=1$, the product is 2.

Assume the product to k is $k+1$.

Answer: At $n=1$, the product is 2. Assume the product to k is $k+1$.

(b) Prove the next case and conclude. [4 marks]

Multiply by $\frac{k+2}{k+1}$.

The product becomes $k+2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

10. A student assumes the result at $k+1$ and rearranges it to the hypothesis at k. Explain the logical gap. [5 marks]

(a) Identify which implication the written argument proves, and which implication induction requires. [3 marks]

Derive $P(k+1)$ from $P(k)$.

If reversing algebra, explicitly justify that each step is reversible.

Answer: The direction written is the reverse of the required implication.

(b) Describe a valid repair of the proof direction. [2 marks]

The required direction is $P(k)$ implies $P(k+1)$.

Answer: Start from the kth hypothesis and derive the successor, justifying every reversal if working backwards.