Identities involving powers and integers

6 questions | 36 marks

Equivalent correct methods earn the corresponding marks.

1. Prove $1+2+2^2+\cdots+2^{n-1}=(2^n-1)$ for all integers $n\geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives $1=1$.

Assume the sum to k is (2^k-1).

Answer: $n=1$ gives $1=1$. Assume the sum to k is (2^k-1).

(b) Prove the next case and conclude. [4 marks]

Add 2^k.

The new total is $2^k-1+2^k=2^{k+1}-1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

2. Prove $1+3+3^2+\cdots+3^{n-1}=\frac{3^n-1}{2}$ for all integers $n\geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

$n=1$ gives $\frac{3-1}{2}=1$.

Assume the sum to k is $\frac{3^k-1}{2}$.

Answer: $n=1$ gives $\frac{3-1}{2}=1$. Assume the sum to k is $\frac{3^k-1}{2}$.

(b) Prove the next case and conclude. [4 marks]

Add 3^k.

The common numerator is $3^k-1+2(3^k)=3^{k+1}-1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $\sum_{r=1}^{n} r\,2^{r-1} = (n-1)2^n + 1$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to k is $(k-1)2^k + 1$.

Answer: At $n = 1$, both sides are 1. Assume the sum to k is $(k-1)2^k + 1$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)2^k$.

The power terms combine to $2k\,2^k = k2^{k+1}$, giving the required formula.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $\prod_{r=1}^{n} \frac{r+1}{r} = n + 1$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the product is 2.

Assume the product to k is $k + 1$.

Answer: At $n = 1$, the product is 2. Assume the product to k is $k + 1$.

(b) Prove the next case and conclude. [4 marks]

Multiply by $\frac{k+2}{k+1}$.

The product becomes $k + 2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $\prod_{r=1}^{n} \frac{2r-1}{2r+1} = \frac{1}{2n+1}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, the product is $\frac{1}{3}$.

Assume the product to k is $\frac{1}{2k+1}$.

Answer: At $n = 1$, the product is $\frac{1}{3}$. Assume the product to k is $\frac{1}{2k+1}$.

(b) Prove the next case and conclude. [4 marks]

Multiply by $\frac{2k+1}{2k+3}$.

The result is $\frac{1}{2k+3} = \frac{1}{2(k+1)+1}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $\sum_{r=1}^{n} (2r-1)^2 = \frac{n(4n^2-1)}{3}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides equal 1.

Assume the formula holds at $n = k$.

Answer: At $n = 1$, both sides equal 1. Assume the formula holds at $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(2k+1)^2$.

The numerator becomes $4k^3 + 12k^2 + 11k + 3 = (k+1)(4k^2 + 8k + 3)$.

This equals $(k+1)[4(k+1)^2 - 1]$, as required.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.