Proofs of summation formulas

6 questions | 36 marks

Equivalent correct methods earn the corresponding marks.

1. Prove $1+2+\cdots+n=\frac{n(n+1)}{2}$ for all integers $n\geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n=1$, both sides are 1.

Assume the sum to arbitrary integer $k\geq 1$ is $\frac{k(k+1)}{2}$.

Answer: At $n=1$, both sides are 1. Assume the sum to arbitrary integer $k\geq 1$ is $\frac{k(k+1)}{2}$.

(b) Prove the next case and conclude. [4 marks]

Add the next term $k+1$.

The next sum is $\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

2. Prove $1+3+\cdots+(2n-1)=n^2$ for all integers $n\geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n=1$, $1=1^2$.

Assume the first k odd numbers sum to k^2.

Answer: At $n=1$, $1=1^2$. Assume the first k odd numbers sum to k^2.

(b) Prove the next case and conclude. [4 marks]

Add the next odd term $2k+1$.

The result is $k^2+2k+1=(k+1)^2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

3. Prove $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $1 = \frac{1(2)(3)}{6}$.

Assume the formula holds at arbitrary $n = k$.

Answer: At $n = 1$, $1 = \frac{1(2)(3)}{6}$. Assume the formula holds at arbitrary $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)^2$ and factor $\frac{k+1}{6}$.

The remaining factor is $k(2k+1) + 6(k+1) = 2k^2 + 7k + 6$.

Factor it as $(k+2)(2k+3)$, giving the required formula at $k+1$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

4. Prove $\sum_{r=1}^{n} r^3 = [\frac{n(n+1)}{2}]^2$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are 1.

Assume the sum to k is $\frac{k^2(k+1)^2}{4}$.

Answer: At $n = 1$, both sides are 1. Assume the sum to k is $\frac{k^2(k+1)^2}{4}$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)^3$ and factor $\frac{(k+1)^2}{4}$.

The remaining factor is $k^2 + 4k + 4 = (k+2)^2$.

The result is $[\frac{(k+1)(k+2)}{2}]^2$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

5. Prove $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, both sides are $\frac{1}{2}$.

Assume the sum to k is $\frac{k}{k+1}$.

Answer: At $n = 1$, both sides are $\frac{1}{2}$. Assume the sum to k is $\frac{k}{k+1}$.

(b) Prove the next case and conclude. [4 marks]

Add $\frac{1}{(k+1)(k+2)}$.

The numerator over the common denominator is $k(k+2)+1 = (k+1)^2$.

Cancellation gives $\frac{k+1}{k+2}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.

6. Prove $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all integers $n \geq 1$. [6 marks]

(a) Verify the first case and state the induction hypothesis. [2 marks]

At $n = 1$, $2 = \frac{1(2)(3)}{3}$.

Assume the formula holds at $n = k$.

Answer: At $n = 1$, $2 = \frac{1(2)(3)}{3}$. Assume the formula holds at $n = k$.

(b) Prove the next case and conclude. [4 marks]

Add $(k+1)(k+2)$.

Factoring gives $\frac{(k+1)(k+2)(k+3)}{3}$.

The true first case and successor implication establish the full domain by mathematical induction.

Answer: The statement holds for all positive integers.