Complex numbers - extended mixed test

10 questions | 60 marks | 75 minutes

Combine exact calculation, constraints, geometric reasoning and factorisation. Later questions require several linked ideas.

Equivalent correct methods earn the corresponding marks.

1. Begin with exact Cartesian calculations. [5 marks]

(a) Simplify $i^{42}+2i^{19}$. [2 marks]

$i^{42}=i^2=-1$ and $i^{19}=i^3=-i$.

Answer: $-1-2i$.

(b) Calculate $(2-i)^2-(1+3i)$. [3 marks]

$(2-i)^2=3-4i$.

Subtract the second bracket.

Answer: $2-7i$.

2. Let $u=\frac{1+2i}{2+i}$. [6 marks]

(a) Find u in Cartesian form. [3 marks]

Multiply numerator/denominator by $2-i$.

The numerator is $4+3i$ and denominator 5.

Answer: $u=\frac{4}{5}+\frac{3}{5}i$.

(b) Show that $u^{-1}=\overline{u}$. [3 marks]

$u\overline{u}=\frac{16+9}{25}=1$.

Thus multiplying u by its conjugate gives its reciprocal condition.

Answer: Both equal $\frac{4}{5}-\frac{3}{5}i$.

3. The polynomial $p(z) = z^2 - (1+3i)z - 4 + 3i$ has root $2+i$. [6 marks]

(a) Find its other root and factorise p. [4 marks]

The sum of roots is $1+3i$.

Subtract $2+i$ to obtain $-1+2i$.

The product is $(2+i)(-1+2i) = -4+3i$.

Answer: Other root $-1+2i$; $p(z) = (z-2-i)(z+1-2i)$.

(b) Explain why the two roots need not be conjugates. [2 marks]

The real-coefficient condition of the conjugate-root theorem is absent.

Answer: The polynomial does not have all-real coefficients.

4. The conjugate roots of a monic real quadratic have midpoint 3 on the Argand diagram and are 8 units apart. [6 marks]

(a) Find the roots. [3 marks]

A conjugate pair has equal real coordinates.

Their midpoint gives real part 3; separation $2|b| = 8$ gives $|b| = 4$.

Answer: $3+4i$ and $3-4i$.

(b) Find the quadratic and the modulus of each root. [3 marks]

The factor product is $(z-3)^2 + 16$.

$|3 \pm 4i| = \sqrt{9+16} = 5$.

Answer: $z^2 - 6z + 25 = 0$; each modulus is 5.

5. For real t, define $w = \frac{t+i}{t-i}$. [7 marks]

(a) Show that $w = \frac{t^2-1}{t^2+1} + \frac{2t}{t^2+1}i$. [3 marks]

The denominator $t - i$ is nonzero for real t.

Multiply by $\frac{t+i}{t+i}$.

The numerator is $(t+i)^2 = t^2 - 1 + 2ti$ and denominator $t^2 + 1$.

Answer: The stated identity holds for every real t.

(b) Show that $|w| = 1$. [2 marks]

$|t+i| = |t-i| = \sqrt{t^2+1} > 0$.

The quotient of these moduli is one.

Answer: Numerator and denominator of the original quotient have equal modulus.

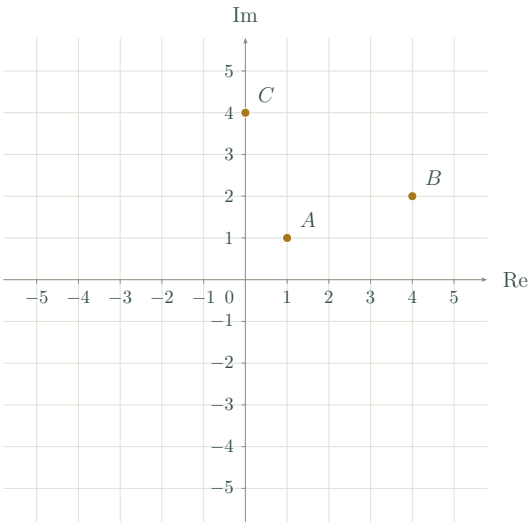
(c) Find t when w is pure imaginary. [2 marks]

The real component is zero when $t^2 - 1 = 0$.

The denominator is positive; the resulting values are i and $-i$.

Answer: $t = 1$ or $t = -1$.

6. A triangle has complex vertices $A = 1 + i$, $B = 4 + 2i$ and $C = 4i$. [6 marks]



(a) Show it is a right-angled isosceles triangle. [4 marks]

The side vectors from A are $(3, 1)$ and $(-1, 3)$.

Each squared length is 10.

The dot product is $3(-1) + 1(3) = 0$.

Answer: $AB = AC = \sqrt{10}$ and $\vec{AB} \cdot \vec{AC} = 0$.

(b) Find its area. [2 marks]

The perpendicular equal sides have length $\sqrt{10}$.

Area $= \frac{1}{2}\sqrt{10}\sqrt{10} = 5$.

Answer: 5 square units.

7. Consider $p(z) = (z^2 - 2z + 5)^2$. [5 marks]

(a) Factor $p(z)$ completely into complex linear factors. [3 marks]

The quadratic has roots $1 \pm 2i$.

Squaring the quadratic repeats each linear factor.

Answer: $(z - 1 - 2i)^2(z - 1 + 2i)^2$.

(b) List the distinct roots and their multiplicities. [2 marks]

Each corresponding factor is squared.

There are four roots when counted with multiplicity.

Answer: $1 + 2i$ and $1 - 2i$, each of multiplicity 2.

8. A supplied transmission model sends z to $w = (1 - 2i)z$. An input is $z = 3 + 4i$. [6 marks]

(a) Find the output and its magnitude. [3 marks]

Expand $(1 - 2i)(3 + 4i) = 11 - 2i$.

$|w| = \sqrt{121 + 4} = \sqrt{125} = 5\sqrt{5}$.

Answer: $w = 11 - 2i$; $|w| = 5\sqrt{5}$.

(b) Find the factor by which this model scales every nonzero input magnitude. [3 marks]

$|w| = |1 - 2i||z| = \sqrt{5}\,|z|$.

The scale factor does not depend on the input.

Answer: $\sqrt{5}$.

9. Let $z = a + bi$ with real a, b. [7 marks]

(a) Show that z^2 has imaginary part $2ab$. [2 marks]

Expand $(a + bi)^2$ and use $i^2 = -1$.

Answer: $z^2 = (a^2 - b^2) + 2abi$.

(b) Classify all z for which z^2 is positive real. [3 marks]

A real square requires $ab = 0$.

If $a = 0$ and $b \neq 0$, then $z^2 = -b^2 < 0$.

Therefore $b = 0$ and $a \neq 0$.

Answer: All nonzero real z.

(c) Classify all z for which z^2 is negative real. [2 marks]

Set $a = 0$, $b \neq 0$ to obtain $-b^2 < 0$.

A nonzero real z would have a positive square.

Answer: All nonzero pure imaginary z.

10. For $z = a + bi$ with $a \neq 0$, define $R = \frac{z - \overline{z}}{z + \overline{z}}$. [6 marks]

(a) Show that $R = \frac{b}{a}i$. [2 marks]

The numerator is $2bi$ and denominator is $2a$.

Cancel the nonzero factor $2a$.

Answer: $R = \frac{bi}{a}$.

(b) Find all z satisfying $R = 2i$ and $|z| = \sqrt{5}$. [4 marks]

The first condition gives $b = 2a$.

The second gives $a^2 + b^2 = 5$, so $5a^2 = 5$.

$a = \pm 1$, with b having the corresponding value ± 2.

Answer: $z = 1 + 2i$ or $z = -1 - 2i$.