Sequences and series - Practice test 5

10 questions | 62 marks | 75 minutes

Integrated sequence reasoning and more demanding proofs.

Equivalent correct methods earn the corresponding marks.

1. An arithmetic sequence satisfies $u_2 + u_6 = 30$ and $S_{10} = 210$. [6 marks]

(a) Find the first term and common difference. [4 marks]

$2a + 6d = 30$ and $2a + 9d = 42$; subtract to get $d = 4$, then $a = 3$.

Answer: $a = 3, d = 4$.

(b) Find u_{20}. [2 marks]

$3 + 19(4) = 79$.

Answer: 79.

2. Three positive geometric terms have sum 26 and product 216. [7 marks]

(a) Show that the middle term is 6. [3 marks]

Write the terms as $\frac{6}{r}, 6, 6r$.

Answer: The product equals the cube of the middle term, so it is $\sqrt[3]{216} = 6$.

(b) Find all possible ordered triples. [4 marks]

$\frac{6}{r} + 6 + 6r = 26$ gives $3r^2 - 10r + 3 = 0$; $r = 3$ or $\frac{1}{3}$.

Answer: $(2, 6, 18)$ and $(18, 6, 2)$.

3. A convergent geometric series has infinite sum 16 and its first two terms sum to 12. [7 marks]

(a) Find all real common ratios. [4 marks]

$a = 16(1 - r)$ and $a(1 + r) = 12$ give $16(1 - r^2) = 12$.

Answer: $r = \frac{1}{2}$ or $-\frac{1}{2}$.

(b) Find the corresponding first terms and explain why both converge. [3 marks]

Use $a = 16(1 - r)$ and check $|r| < 1$.

Answer: $a = 8$ or 24, respectively; both ratios have magnitude below 1.

4. $u_0 = 2$ and $u_{n+1} = u_n^2$. [5 marks]

(a) Find u_1, u_2, u_3. [2 marks]

Square each preceding result.

Answer: $4, 16, 256$.

(b) Give an explicit rule and explain why this is not an ordinary geometric sequence. [3 marks]

Each squaring doubles the exponent; the ratios begin $2, 4, 16$.

Answer: $u_n = 2^{2^n}$; consecutive ratios are not constant.

5. A conservation model begins at 50 animals and updates by retaining 90% then adding 8 animals. [6 marks]

(a) Find its explicit rule and value after two steps. [3 marks]

$L = \frac{8}{0.1} = 80$; the initial deviation is -30.

Answer: $u_n = 80 - 30(0.9)^n$; $u_2 = 55.7$.

(b) Explain why 80 is never attained at a finite step in this model. [3 marks]

Values approach 80 from below.

Answer: The negative deviation $-30(0.9)^n$ is never zero at finite n.

6. End-of-year deposits are AUD 100, AUD 110 and AUD 121 in years 1, 2 and 3. Interest is 10% yearly. [5 marks]

(a) Find the three contributions immediately after the third deposit. [2 marks]

They are $100(1.1)^2$, $110(1.1)$ and 121.

Answer: AUD 121 from each deposit.

(b) Find the balance and explain why the usual equal-deposit formula cannot be applied directly. [3 marks]

The separate contributions happen to coincide because deposits and interest use the same growth factor.

Answer: 363; the payment amounts are not equal.

7. A ball drops 5 m and rebounds to 60% of the previous height each time. [7 marks]

(a) Find the ideal total vertical distance. [3 marks]

$5 + 2\left[\frac{3}{1-0.6}\right] = 20.$

Answer: 20 m.

(b) Find the total distance by the time it has returned to the ground after its second rebound. [2 marks]

$5 + 2(3) + 2(1.8) = 14.6.$

Answer: 14.6 m.

(c) Find the remaining distance in the infinite model. [2 marks]

$20 - 14.6 = 5.4.$

Answer: 5.4 m.

8. For fixed a and $r \neq 1$, prove $S_n = \frac{a(1-r^n)}{1-r}$. [7 marks]

(a) Check the base and state the hypothesis. [2 marks]

At $n = 1$, cancellation gives a.

Answer: $S_1 = a$; assume $S_k = \frac{a(1-r^k)}{1-r}$.

(b) Prove the next case and state how $r = 1$ is handled. [5 marks]

Add ar^k. The combined numerator is $a[1 - r^k + r^k(1-r)] = a(1-r^{k+1})$. Conclude by induction.

Answer: $S_{k+1} = \frac{a(1-r^{k+1})}{1-r}$; for $r = 1$, $S_n = na$.

9. A proposed identity is $1 + 2 + \cdots + n = \frac{n(n+1)}{2} + 7$. [6 marks]

(a) Check its base at $n = 1$. [2 marks]

The claimed identity fails immediately.

Answer: False: 1 is not 8.

(b) Show that assuming its k case would nevertheless lead formally to the same wrong $k+1$ formula. Explain the lesson. [4 marks]

An implication from a false premise does not establish any true starting point.

Answer: $\frac{k(k+1)}{2} + 7 + (k+1) = \frac{(k+1)(k+2)}{2} + 7$. The base is indispensable.

10. A model uses $u_n = 100(1.04)^n$ at whole-number years $n \geq 0$. [6 marks]

(a) Find the first whole year when $u_n > 200$. [3 marks]

$1.04^{17} \approx 1.9479 < 2$ and $1.04^{18} \approx 2.0258 > 2$.

Answer: Year 18.

(b) Explain why rounding the real solution of $100(1.04)^n = 200$ to the nearest integer is not the correct general method. [3 marks]

Nearest-integer rounding may produce a term still below the threshold.

Answer: The first exceeding index must be the next allowed integer above the threshold, then checked.